Topic 4: Special Relativity and Thomas Precession

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CSUEB Physics
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If you can’t explain it simply, you don’t understand it well enough.

Albert Einstein (1879-1955)

1905 Special Relativity
Index: Rough Draft

A. Lorentz Transformations

B. Velocity Addition

C. Acceleration & Thomas Precession

D. References

A. Lorentz Transformations

1. Special Relativity Effects

2. Lorentz Transformation

3. Clifford Algebra form of LT
A1. Lorentz-FitzGerald Contraction

1889 FitzGerald, 1892 Lorentz

• Propose a moving meter stick will appear to shrink in length

\[ L' = L \sqrt{1 - \left(\frac{v}{c}\right)^2} \]

• 1905 Einstein deduces this from his postulates of relativity.

A2. Time Dilation

Proposed 1897 by Larmor
1904 by Lorentz

• A moving clock will appear to run slower

\[ t' = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}} \]

Paradoxically, two observers each moving with a clock, sees the OTHER clock running slowly.
A.3 Special Relativity

1905 Einstein postulates:

- Motion is relative
- Speed of light is same for all observers

From these postulates he recovers Lorentz’s transformations

Lorentz Transformation (1904)

- 1904 Lorentz, 1905 Poincare & Einstein
- Motion is equivalent to a hyperbolic rotation in spacetime
- Angle $\beta$ is the “rapidity”: $\tanh \beta = v/c$

\[
\begin{pmatrix}
ct' \\
x' \\
y'
\end{pmatrix} =
\begin{pmatrix}
\cosh \beta & \sinh \beta & 0 \\
\sinh \beta & \cosh \beta & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
ct \\
x \\
y
\end{pmatrix}
\]

\[
\cosh \beta = \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}
\]
Clifford Algebra Form of Rotation

• Rotation of any quantity “Q” in “xy” plane by angle $\phi$ can be expressed in exponential “half angle” form:

$$Q' = R \, Q \, R^{-1}$$

$$R = e^{\hat{e}_1 \hat{e}_2 \phi/2} = \cos \frac{\phi}{2} + \hat{e}_1 \hat{e}_2 \sin \frac{\phi}{2}$$

Note “spacelike” bivectors square negative:

$$(\hat{e}_1 \hat{e}_2)^2 = -1$$

Clifford Algebra Form of LT

• Hyperbolic Rotation in “TX” plane by rapidity angle $\beta$ has same exponential “half angle” form:

$$Q' = L \, Q \, L^{-1}$$

$$L = e^{\hat{e}_4 \hat{e}_1 \beta/2} = \cosh \frac{\beta}{2} + \hat{e}_4 \hat{e}_1 \sinh \frac{\beta}{2}$$

Note “Timelike” bivectors square positive:

$$(\hat{e}_4 \hat{e}_1)^2 = +1$$

because

$$(\hat{e}_4)^2 = -1$$

$$(\hat{e}_1)^2 = +1$$
Hyperbolic Geometry Review

• Spacetime is Hyperbolic, not trigonometric

\[
\sinh \beta = \frac{1}{2} \left( e^\beta - e^{-\beta} \right) \\
\cosh \beta = \frac{1}{2} \left( e^\beta + e^{-\beta} \right) \\
\tanh = \frac{\sinh}{\cosh} \\
\cosh^2 - \sinh^2 = -1
\]

B. Velocity Addition Formula

• Velocities don’t add: \( V \neq V_1 + V_2 \)

• Rapidities DO add: \( \beta = \beta_1 + \beta_2 \)
**B.1 Addition of Velocities**

- Speeds add: \[ V = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \]

- Adding anything to speed of light gives speed of light

Incorrect Newtonian description:
As seen by astronaut in spaceship, light is approaching her at
\[(3 \times 10^8 \text{ m/s}) + (1 \times 10^8 \text{ m/s}) = 4 \times 10^8 \text{ m/s} \]

Correct Einsteinian description:
As seen by astronaut in spaceship, light is approaching her at
\[3 \times 10^8 \text{ m/s} \]

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**Parallel Velocity Addition (Einstein 1905)**

- Rapidity angles add, not velocities

\[ \tanh(\beta_1 + \beta_2) = \frac{\tanh \beta_1 + \tanh \beta_2}{1 + \tanh \beta_1 \tanh \beta_2} \]

\[ \tanh \beta = \frac{V}{c} \]

\[ V = \frac{V_1 + V_2}{1 + \frac{V_1 V_2}{c^2}} \]
velocity composition

- the general addition of velocities in two different directions is neither commutative or associative:

\[
u \oplus v = \frac{1}{1 + \frac{u \cdot v}{c^2}} \left[ u + v + \frac{1}{c^2} \gamma (u \times (u \times v)) \right]
\]

\[
\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}
\]

thomas-wigner rotation 1939

- 1775 euler: the composition of two rotations is a single rotation.
- wigner shows that in general the composition of two lorentz transformations is a lorentz transformation plus a rotation
- why isn’t it just a lorentz transformation?

spacelike bivectors are closed under multiplication

\[
\begin{align*}
(e_1, e_2)(e_2, e_3) &= e_1 e_3 \\
(e_2, e_3)(e_3, e_1) &= e_2 e_1 \\
(e_3, e_1)(e_1, e_2) &= e_3 e_2
\end{align*}
\]

timelike bivectors are not closed under multiplication, they yield rotations

\[
\begin{align*}
(e_4, e_1)(e_4, e_2) &= e_1 e_2 \\
(e_4, e_2)(e_4, e_3) &= e_2 e_3 \\
(e_4, e_3)(e_4, e_1) &= e_3 e_1
\end{align*}
\]
Derivation using Half Angles

- I have not seen this done anywhere, but possibly Abraham A. Ungar has it in his book somewhere.

- Want to show:
  \[ e^{\frac{\alpha}{2}} e^{\frac{\beta}{2}} = e^{\frac{\gamma}{2}} e^{\frac{\theta}{2}} \]

- where \{\alpha, \beta, \gamma\} are timelike bivectors for Lorentz

- \( \alpha^2 = \beta^2 = \gamma^2 = +1 \)

- \( \theta \) is spacelike bivector of rotation

- \( \theta^2 = -1 \)

- Expand:

\[
(cosh\frac{\alpha}{2} + \hat{\alpha} sinh\frac{\alpha}{2})(cosh\frac{\beta}{2} + \hat{\beta} sinh\frac{\beta}{2}) = (cosh\frac{\gamma}{2} + \hat{\gamma} sinh\frac{\gamma}{2})(cosh\frac{\theta}{2} + \hat{\theta} sinh\frac{\theta}{2})
\]

Solution of Wigner Rotation

- Solution is analogous to the Rodrigues formula!

- Net Velocity is symmetric!

\[
\gamma \tan \frac{\gamma}{2} = \frac{\hat{\alpha} \tan \frac{\alpha}{2} + \hat{\beta} \tan \frac{\beta}{2}}{1 + \hat{\alpha} \cdot \hat{\beta} \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}
\]

- Rotation

\[
\theta \tan \frac{\theta}{2} = \frac{\hat{\alpha} \otimes \hat{\beta} \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}{1 + \hat{\alpha} \cdot \hat{\beta} \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}
\]
C. Thomas Precession

Llewellyn Hilleth Thomas (1903-1992)

• 1926 proposes “Thomas Precession” to explain the difference between predictions made by spin-orbit coupling theory and experimental observations.

Thomas Precession in Brief

• Acceleration perpendicular to velocity causes an extra rotation of the reference frame, with angular velocity:

\[ \omega_{th} \approx \frac{1}{2c^2} \vec{a} \times \vec{v} \]

• The change in any vector in lab frame is related to change in rotating (due to accelerating) body frame by:

\[ \frac{d\vec{V}}{d\tau}_{\text{lab}} = \frac{d\vec{V}}{d\tau}_{\text{body}} + \vec{\alpha}_h \times \vec{V} \]
Thomas Precession in Orbits

• Atomic: After electron completes one closed orbit (in period “T”), there will be an “anomalous” net rotation of the (spin axis of the) electron due to the acceleration being perpendicular to the velocity.

\[ \Delta \theta \approx \omega_{th} T = 2\pi \frac{v^2}{c^2} \]

• This is also true for any circular orbit, for example a Foucault pendulum, which precesses at rate of 222° a day at Hayward, would have an additional Thomas precession due to rotation of earth on the order of:

\[ \Delta \theta \approx 1.5 \times 10^{-11} \text{ rads/day} = 3 \times 10^{-6} \text{ arc sec/day} \]

1. Fermi-Walker Transport

• Basis vectors \( e_k \) fixed to an accelerating NON-rotating rigid body will change according to:

\[ \dot{e}_k = \frac{1}{c^2} (u \wedge a) \cdot e_k = \frac{1}{c^2} (a_k u - u_k a) \]

– 4 velocity “u”
– 4 acceleration: \( a = \ddot{u} \)

• The change in any vector in lab frame is related to change in (accelerating) body frame by:

\[ \frac{d\vec{V}}{d\tau_{\text{lab}}} = \frac{d\vec{V}}{d\tau_{\text{body}}} + \frac{1}{c^2} (u \wedge a) \cdot \vec{V} \]
Clifford Algebra derivation

- Lorentz Transformation of basis vector (at time “t”)
  \[ e_k(t) = L e_j(0) L^{-1} \]

- Operator for boost in direction of velocity described by rapidity \( \beta \)
  \[ L = e^{\hat{\beta} \beta / 2} \]

- I think you can show that the Fermi-Walker transport bivector is given:
  \[ \frac{1}{c^2} u \wedge a \equiv 2 \dot{L} R^{-1} = -2 \dot{L} \dot{L}^{-1} \]

Incomplete derivation

- Differentiating exponential operator I get:
  \[ 2 \dot{L} L^{-1} = \dot{\beta} \hat{\beta} + \frac{d\dot{\beta}}{d\tau} \sinh \beta + (1 - \cosh \beta) \hat{\beta} \otimes \frac{d\dot{\beta}}{d\tau} \]

- The last term is the Thomas Precession

- I haven’t yet shown this gives exactly
  \[ \frac{1}{c^2} u \wedge a \equiv 2 \dot{L} R^{-1} = -2 \dot{L} \dot{L}^{-1} \]
Thomas Term from Fermi-Walker

- Expanding the 4-vectors one gets
  \[ u \wedge a \approx \gamma^3 \vec{v} \wedge \vec{a} + \gamma^3 \gamma \vec{e}_4 e_k \vec{a}^k \]
- The last term is simply the linear acceleration (timelike bivector)
- The first term is a spacelike bivector, i.e. a rotational angular velocity
  \[ \vec{\omega}_{th} = \frac{1}{c^2} \gamma^3 \vec{v} \times \vec{a} \]
- Ouch, not quite right. Should be:
  \[ \vec{\omega}_{th} = \frac{1}{c^2} \gamma^2 \frac{\vec{a} \times \vec{v}}{\gamma + 1} \]

Larmor Precession in Atomic Orbits

- An electron, with magnetic moment \( \mu \) which is moving with constant velocity in electromagnetic field will have a torque on it:
  \[ \vec{S} = \vec{\mu} \times (\vec{B} - \frac{1}{c^2} \vec{v} \times \vec{E}) \]
  \[ e\vec{E} = m\vec{a} \]
  \[ \vec{\mu} \approx \frac{e}{m} \vec{S} \]
- The magnetic moment is proportional to the spin. Substitute this for \( \mu \)
- We get precession rate:
  \[ \omega_{\text{Larmor}} = \frac{\vec{v} \times \vec{a}}{c^2} \]
  \[ \dot{S} = \omega_{\text{Larmor}} \times S \]
Larmor plus Thomas

- Note the Larmor precession is just double the Thomas, and with opposite sign.

\[ \omega_{\text{Larmor}} = +\frac{\vec{v} \times \vec{a}}{c^2} = -2\omega_{\text{th}} \]

- The total precession is hence the algebraic sum

\[ \omega_{\text{net}} = \omega_{\text{Larmor}} + \omega_{\text{th}} = \frac{1}{2} \frac{\vec{v} \times \vec{a}}{c^2} \]

- Thus, observed precession is half the expected Larmor precession (“The Thomas half”)

References

- Baylis, Electrodyanmics, has a coherent description of Thomas Precession, using 3D Clifford algebra (i.e. Pauli Algebra). The Wigner rotation is derived in equation (4.29). Thomas Precession in section 4.6.
- L. H. Thomas, “Motion of the spinning electron”, Nature 117, 514, (1926)
- Tomonaga, The Story of Spin, Lecture 2 and 11.
Hmk Ideas

- Calculate the Thomas precession rate for the moon. How many years would it take to have the side of the moon facing us change by 180 degrees?
- Calculate the Thomas precession rate for a point on the equator of the earth
- Does the Kimbal experiment measure Thomas precession due to earth’s rotation? (this is separate from the Foucault precession)
Things to do

- Obviously need to clean up the derivation of Thomas precession term.