Rotational Motion and Moment of Inertia

Purpose:
To determine the rotational inertia of a disc and of a ring and to compare these with the theoretical values.

Equipment:
- Rotating Table, Disc, Ring
- Hooked Mass Set
- Long Rod
- Right Angle Clamp
- Cylindrical Rod Clamp
- Table Clamp
- Smart Pulley/Photogate
- Stop Watch
- 2-meter Stick
- Vernier Calipers
- Carpenter’s Level
- String

I. Theory:
By now you’ve probably figured out that every variable that was defined when talking about linear motion has an analogue in rotational motion. Instead of distance traveled, d, we have angle turned, θ. Rather than speaking simply of velocity, v, we talk of angular velocity, ω. Every aspect of linear motion has its partner in rotational motion. And not only are they analogous, but they’re related, too! …usually by some power of the radius of the circle.
In fact, here’s how it goes…
### Linear Variable | Symbol | Angular Variable | Symbol | Relationship
--- | --- | --- | --- | ---
Distance | d | Angle | \( \theta \) | \( d = \theta \cdot r \)
Velocity | \( v \) | Angular velocity | \( \omega \) | \( v = \omega \cdot r \)
Acceleration | \( a \) | Angular acceleration | \( \alpha \) | \( a = \alpha \cdot r \)
Force | \( F \) | Torque | \( \tau \) | \( \tau = r \times F \)
Mass | \( m \) | Rotational inertia | \( I \) | \( I = \int r^2 \, dm \)
Momentum | \( p \) | Angular momentum | \( L \) | \( L = r \times p \)

**Figure 1** Relationships between Linear and Angular (Rotational) Variables

The same basic equations of linear motion can be used for rotational motion with a simple change of variables:

\[
\begin{align*}
x_f &= x_i + v_i \cdot t + \frac{1}{2} a \cdot t^2 \\
v_f &= v_i + a \cdot t
\end{align*}
\]

\[
\begin{align*}
\theta_f &= \theta_i + \omega_i \cdot t + \frac{1}{2} \alpha \cdot t^2 \\
\omega_f &= \omega_i + \alpha \cdot t
\end{align*}
\]

**Eq. 1**

The difference between angular and linear motion, and the relation between them, is something that most students are able to understand fairly easily. Almost everyone has played on a playground merry-go-round, and has noticed the difference in velocity between standing at the center and standing at the edge. The confusion begins to set in with discussions of torque and rotational inertia, the questions being specifically why do we need to differentiate between the simple Force and Mass?

When discussing linear motion, it is assumed that all objects are “point objects” whose mass is centered symmetrically about a single point, and they have no size to speak of – that is, all objects are shaped essentially like very small balls that don’t spin. Most objects, however, don’t actually fit this category. Most objects (even most balls) are *extended objects* – they have a measurable size and they may not be totally symmetric. When a force is exerted on an extended object, it matters not only how big the force is, but also where on the object it is applied. Imagine a bar on a table:

**Figure 2** The application of a force, and the resulting motion of a bar
If the force is applied directly to the center of the object, it will translate linearly across the table. However, a push on either side of the center will cause a rotation of the object.

When a force is applied away from the axis of rotation (in this case, the center of mass) it causes the object to rotate. This is what we call a \textit{torque}. The torque is defined as

\[ \tau = r \times F \]  

\text{Eq. 2} \]

where $F$ is the force applied and $r$ is the distance from the axis of rotation to where the force is applied. Remember that the definition of a cross product is:

\[ A \times B = |A||B| \sin \theta \]

where $\theta$ is the angle between vectors $A$ and $B$. Thus,

\[ \tau = |r||F|\sin \theta \]  

\text{Eq. 3} \]

This means that the largest torque occurs when $r$ and $F$ are perpendicular to each other ($\sin \theta = 1$) and the smallest torque is when $r$ and $F$ are parallel ($\sin \theta = 0$). As an example, think of using a wrench to turn a bolt:

- **Largest torque**
  \[ \tau = |r||F|\sin \theta \]
  where $\sin \theta = 1$
  so $\tau = |r||F|$

- **Smaller torque**
  \[ \tau = |r||F|\sin \theta \]
  where $\sin \theta < 1$
  so $\tau < |r||F|$

- **Smallest torque**
  \[ \tau = |r||F|\sin \theta \]
  where $\sin \theta = 0$
  so $\tau = 0$

**Figure 3 Torque: Changing the Direction of the Applied Force**

Similarly, we can change the radius of rotation:

- **Largest $r$ → largest $\tau$**
- **Smallest $r$ → smallest $\tau$**

**Figure 4 Torque: Changing the Radius of Rotation**

If you’ve ever tried to turn a tightened bolt, you probably already know that the best results come when the force is applied to the end of the handle, in a perpendicular direction.

What then, about rotational inertia?

Remember Newton’s First Law of Motion, sometimes called “The Law of Inertia”:

\begin{quote}
\textbf{Any object will continue in its state of rest or straight-line motion unless acted upon by an outside force.}
\end{quote}
“Any object” really means “any object with mass”, which is usually a safe assumption. Practically, this means that any object that has mass resists changes in its motion. This resistance is what is referred to as inertia. Since inertia is a property of all objects having mass, some go so far as to say that inertia is mass (or, mass is inertia).

Now think back to the wrench again. Imagine holding one end in your hand and rotating your wrist to swing the other end in a small arc. Now switch ends. Is it easier one way than the other? It should be! You should find that it is easier to swing the wrench when you are holding the rounded (clamping) end. It is harder when you allow that same (rounded) end to swing.

NOTE: If you have never actually done this, or have forgotten what it’s like, ask your lab instructor or lab tech for a wrench to try it out.

Why would this be? The wrench has the same mass no matter how you hold it, and if you exert the same force, it seems that it should accelerate in the same manner. You have just discovered the reason for defining an object’s Rotational Mass!!

Rotational inertia takes into account not only the total mass of an object, but how the mass is distributed around an axis of rotation, and just as mass tells us how much resistance a force will meet, rotational mass (or moment of inertia) tells us how much an object will resist rotating (or changing its rotation). A single object’s rotational inertia will change depending on where the pivot point (or axis) is placed. It is easier to turn an object when most of the mass is located near the pivot point, thus this configuration has a lower rotational inertia. If most of an object’s mass is located far away from the pivot point, the rotational inertia is larger, and it will be more difficult to rotate.

Rotational mass can be defined in two (equivalent) ways:

\[ I = \int mr^2 dV \quad \text{a} \]
\[ I = \int r^2 dm \quad \text{b} \]

The expression you use depends mostly on what proves easiest for the particular situation. Most often, you will not need either, for the geometry of the system will be simple enough to use some combination of the pre-defined moments of inertia.
When we want to quantify the relation between the force exerted on an object and its mass (or inertia) we use Newton’s Second Law of Motion:

\[ F = ma \]  \hspace{1cm} \text{Eq. 5}

To quantify the relation between the torque on an object and its rotational mass (or inertia), we simply replace each part of Newton’s Second Law with its rotational analogue:

\[ \tau = I \alpha \]  \hspace{1cm} \text{Eq. 6}

Thus, we can perform an experimental test. The equations in the chart above allow calculation of the theoretical rotational mass of a particular disk or ring. By applying a known torque to the object and measuring the angular acceleration, we can determine an experimental rotational mass, and compare the two.
II. Experiment

A. Setup & Calibration

1. Set up equipment as shown in the sketch below. Let \( m = 200 \text{ g} \).

   ![Figure 6 Basic Experimental Set-Up](image)

   - **Question 1:** What is the radius of the drum “\( r \)” and its absolute and percent uncertainty? [Note, consider whether the string winds over itself on the drum which would change the radius during the experiment]

2. Acceleration Calibration Test: For the first run, \((m = 200 \text{ g}, \text{empty table})\) measure the distance, \( h \) the weight falls and the time, \( t \), (using a stopwatch) it takes the weight to fall that distance. Determine the acceleration from the relation: \( h = \frac{1}{2} at^2 \).

3. LabPro Setup:
   - Connect the AC adapter to the LabPro by inserting the round plug on the 6-volt power supply into the side of the interface. Shortly after plugging the power supply into the outlet, the interface will run through a self-test. You will hear a series of beeps and blinking lights (red, yellow, then green) indicating a successful startup.
   - Attach the LabPro to the computer using the USB cable that is Velcro-ed to the side of the computer box (do not unplug the USB cable from the computer!). The LabPro computer connection is located on the right side of the interface. Slide the door on the computer connection to the right and plug the square end of the USB cable into the LabPro USB connection.
   - Connect the Smart Pulley/Photogate to the DIG/SONIC1 port of the LabPro. If you are using a one-piece Smart Pulley/Photogate, a PASCO or very old Vernier Photogate, you will need to use the digital adapter. If you are using a newer
4. **LabPro Software Template:**

- Open the file `rotational_inertia.xmbl` (or `.cmbl`) in the **Experiments** folder on the desktop. This will start the program Logger Pro3.3 and bring up the appropriate data file. If you do not have an auto-ID sensor (which is the likely case), a dialog box will pop up asking you to confirm the sensors being used. If you have the suggested sensor attached to the LabPro in the suggested port, click “OK”. If the “OK” button is not active, ask your instructor for help.

- Once Logger Pro 3 is open, click on **Experiment > Set Up Sensors > LabPro 1**.
  
  **Click on the photogate icons, and verify that “Motion Timing” is selected under “Current Calibrations”.** In the same menu, choose “Set Distance or Length…”, make sure that “Smart Pulley (10 Spoke) in Groove” is selected. The program calculates the acceleration and velocity of the falling mass by treating the pulley as a picket fence with the “proper” spacing.

- Now press **Collect** and release the mass. Check the results of the computer measurements against your own measurements of accelerations (from height and time). The "by hand" values can differ by as much as 20% from the Logger Pro values.

**B. Experiments**

1. **Configurations:** We will measure the moment of inertia for following configurations (each will be a separate graph)
   - Disk (plus Cradle)
   - Ring (plus Cradle)
   - Empty cradle
   - Disk plus Ring (plus Cradle)

2. **Data to Take for Each Configuration**
   - For a given hung mass, measure the acceleration of the system (using Logger Pro).
     - You can take the average of the accelerations displayed by Logger Pro
     - Or, you can take the velocity data from Logger Pro (and the time data), and plot Velocity vs Time in Excel, apply a Trend Line to get the slope (i.e. the acceleration). This probably would give a better value!
     - Make sure you are only using the data corresponding to constant acceleration, not the data after the weight hits the ground etc.
   - Repeat: measure the acceleration for at least 5 different hung masses.
3. Make a plot of String Tension vs acceleration (for each configuration)

(a) Theory Discussion: How to correct for frictional effects.

- Equation of motion for accelerating hung mass: \( mg - T = ma \)
- Solve for tension: \( T = m(g-a) \)
- We know “m”, and we are directly measuring “a”, so we can calculate tension “T”.
- Torque on drum (radius \( r \)) is: \( \tau = r(T-f) \), where “f” is unknown frictional forces
- Angular acceleration of apparatus: \( \tau = I \alpha \)
- Since string unrolls from drum: \( a = r \alpha \)
- Hence torque equation becomes (solving for T): 
  \[
  T = \left( \frac{I}{r^2} \right) a + f
  \]
- Hence a plot of tension \([T=m(g-a)]\) vs acceleration “a” should give a line with intercept giving the friction, and the slope giving the moment of inertia of the system divided by squared \( r \).

(b) Make Plot of Tension vs acceleration

- Tension: \( T = m(g-a) \), where “m” is the hung mass and “a” is measured acceleration
- Fit the best line to the points
- The slope of the line is the moment of inertia (divided by squared radius of drum), while the intercept is the frictional force in the system.
- If you used the prepared Excel template, it will also calculate the percent uncertainties in the slope and intercept (details of this calculation provided in appendix)

Question 2: For each plot

(a) Are your plots lines as expected (i.e. validating theory that friction is constant)?
(b) What are the R-squared for each plot?
(c) Are the intercepts (friction) nearly the same, or does it show that there is more friction when the turntable is more heavily loaded?

Question 3: For each plot (suggest a tabular display of this information)

(a) What is your slope (and its uncertainty)?
(b) Calculate the measured moment of inertia: \( I = (slope) \times r^2 \)
(c) Calculate the percent uncertainty in the moment of inertia: 
  \[
  e_I = \sqrt{e_{slope}^2 + 4e_r^2}
  \]
  [Note I have used “e” as the symbol for percent uncertainty]
Sample Data & Graph for Configuration: Cradle

<table>
<thead>
<tr>
<th>Config</th>
<th>Mass (kg)</th>
<th>Accel (m/s²)</th>
<th>Tension (N)</th>
<th>Item</th>
<th>units</th>
<th>Value</th>
<th>% Unc</th>
</tr>
</thead>
<tbody>
<tr>
<td>Empty</td>
<td>0.050</td>
<td>0.034</td>
<td>0.488</td>
<td>Radius</td>
<td>meter</td>
<td>0.03</td>
<td>3.0%</td>
</tr>
<tr>
<td>Empty</td>
<td>0.100</td>
<td>0.130</td>
<td>0.967</td>
<td>Slope</td>
<td>kg</td>
<td>5.411</td>
<td>2.0%</td>
</tr>
<tr>
<td>Empty</td>
<td>0.200</td>
<td>0.296</td>
<td>1.901</td>
<td>M. Inertia</td>
<td>kg·m²</td>
<td>0.00487</td>
<td>6.3%</td>
</tr>
<tr>
<td>Empty</td>
<td>0.300</td>
<td>0.437</td>
<td>2.809</td>
<td>Friction</td>
<td>N</td>
<td>0.312</td>
<td>6.7%</td>
</tr>
<tr>
<td>Empty</td>
<td>0.400</td>
<td>0.624</td>
<td>3.670</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Empty</td>
<td>0.500</td>
<td>0.784</td>
<td>4.508</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>Empty</td>
<td></td>
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</tr>
</tbody>
</table>

You can see the measured moment of inertia (from slope) is 0.00487 kg·m².

From the intercept, the friction measured is 0.342 Newtons.

Obviously the data displays a very good linear fit!
C. Analysis

1. **Theoretical Moments of Inertia:**
   - Measure mass of disk, and its radius (be sure to estimate uncertainty in radius).
   - Calculate the moment of inertia:  \( I = \frac{1}{2} MR^2 \)
   - Similarly, calculate the moment of inertia of the disk:  \( I = \frac{1}{2} M \left( R_2^2 - R_1^2 \right) \)

   **Question 4: Theoretical Moments of Inertia**
   (a) Summarize your measured parameters of the disk and its theoretical moment of inertia.
   (b) Do the same for the ring.
   (c) Approximately what is the uncertainty in your measurements of these quantities?

2. **Measured Moments of Inertia:**
   - Subtract the moment of inertia of the empty cradle from the moment of inertia of disk plus cradle to get just the moment of inertia of the disk. Do the same for the ring.
   - Calculate the uncertainties of the moment of inertia of the disk (and ring)
     - First convert the percent uncertainties in the measured moments of inertia to absolute uncertainties
     - Recall that when you add (or subtract) quantities, the absolute uncertainty of the sum (or difference) is the square root of the sums of the squares of the absolute uncertainties of the components.
     - Reconvert the absolute uncertainty of the disk (or ring) back into a percentage

   **Question 5: Measured Moments of Inertia**
   (a) Summarize your measured moment of inertia for the disk (and its uncertainty).
   (b) Do the same for the ring.
   (c) Are the uncertainties for these “measured” inertias bigger or smaller than for the “theoretical” values?

   **Question 6: Measured Moments of Inertia**
   If you had time to measure the moment of inertia of the ring+disk+cradle then:
   (a) Calculate the moment of inertia of disk from:  \( (\text{ring+disk+cradle})-(\text{ring+cradle}) \)
   (b) Calculate the moment of inertia of ring from:  \( (\text{ring+disk+cradle})-(\text{disk+cradle}) \)
   (c) Compare these answers with question 5ab. Are they close? (within uncertainty?)

3. **Compare Theory with Measurement:** (consider a tabular display of this info)
   - Compare your measured values of moment of inertia to the theoretical values.
   - Compute the error
     - absolute error = measured – theoretical
     - percent error = \( \frac{(\text{measured/ theoretical} -1)}{1} \times 100\% \)
   - If your error is smaller than the uncertainty you have a good result. If the error is much bigger than the uncertainty then the probability is that you made a poor measurement.

   **Question 7: Comparison**
   Summarize your comparison of theory to measured values of moment of inertia. Are the differences within uncertainty?