Poly-Dimensional Relativity, A Classical Generalization of Crawford's Automorphism Invariance Principle

Presentor: William M. Pezzaglia Jr.

(wpezzag@clifford.org)

Department of Physics, Santa Clara University, Santa Clara, CA 95053, USA

Pesentation at: 4 th Internationa Conference on Clifford Algebras and their Applications in Mathematical Physics, Aachen, Germany, May 1996

Abstract

The automorphism invariant theory of Crawford [J. Math. Phys. 35, 2701 (1994)] has shown great promise, however its application is limited by the paradigm to the domain of spin space. Our conjecture is that there is a broader principle at work which applies even to classical physics. Specifically, the laws of physics should be invariant under polydimensional transformations which reshuffle the geometry (e.g. exchanges vectors for trivectors) but preserves the algebra. To complete the symmetry, it follows that the laws of physics must be themselves polydimensional, having scalar, vector, bivector etc. parts in one multivector equation. Clifford algebra is the natural language in which to formulate this principle, as vectors/tensors were for relativity. This allows for a new treatment of the relativistic spinning particle (the Papapetrou equations) which is problematic in standard theory. In curved space the rank of the geometry will change under parallel transport, yielding a new basis for Weyl's connection and a natural coupling between linear and spinning motion.

Note: this talk was summarized in: **Clifford Algebras and their Applications in Mathematical Physics** (Proceedings of 4th Conference, Aachen, Germany 1996), Dietrich, Habetha and Jank (eds.), Kluwer (1998), pp. 305-317. Preprint available at gr-qc/9608052.

- index of other talks at: http://www.clifford.org/~wpezzag/talks.html
- This URL: **http://www.clifford.org/wpezzag/talk/96aachen/**
  Updated: 2005May20

Index to Transparencies Color and BW.

Poly-Dimensional Relativity

A Classical Generalization of (Jim Crawford's) Automorphism Invariance Principle

William M. Pezzaglia Jr.
Dept of Physics
Santa Clara University
Santa Clara, California, USA

May 29, 1996
Aachen, Germany

I Outline

II. Motivation / Introduction

III Extend Special Relativity

A. Review Standard
B. Clifford Space, generalized Coords
C. Polydim Mechanics (spinning Particle)

IV Extend General Relativity

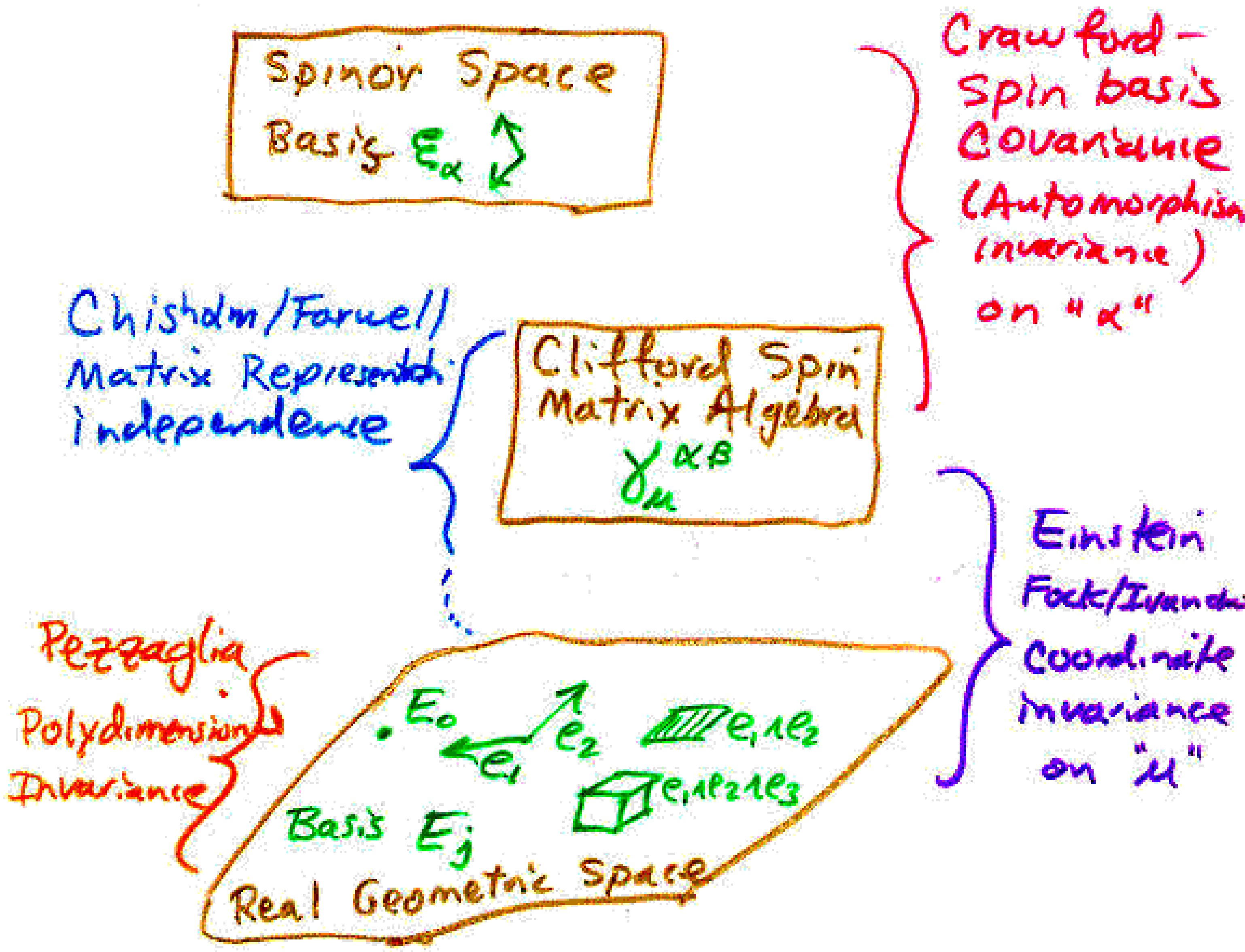
A. Review Standard.
B. Curvature Changes type of Geometry
C. Polygeometrodynamics (particle in space)

V Epagogical Summary

New principles

II Introduction / Motivation (Rationalization)

New physics comes from generalized principles



- Chisholm/Farwell: e_μ tied to subset of $\gamma_\mu^{\alpha\beta}$. Higher Dimensions. Rank of e_μ unchanged. $e_\mu = \xi_\alpha \gamma_\mu^{\alpha\beta} \bar{\xi}_\beta$
- Crawford: No connection of γ_μ to e_μ. γ_μ completely reshuffled
- Pezzaglia: No spin space (ξ is left ideal of E_j). Real geometry reshuffled

3.1

III Extension of Special Relativity

A Review some Features of Special Relativity

1 Minkowski Spacetime

(a) Time is the 4th Dimension
Speed of light "C" is same for all observers

(b) Laws of Physics are invariant under Lorentz Transformations (same in all non-accelerated frames) motion is relative

(c) Invariant Length $d\Sigma^2 = c^2 d\tau^2$
Proper Time "τ" $= c^2dt^2 - (dx^2+dy^2+dz^2)$

2. Classical (Lagrangian) Mechanics

(a) Principle of Least Action:
Particles follow shortest Path

(b) Action $= \int m_0 c\, d\Sigma = \int m_0 c \sqrt{\dot{x}_\mu \dot{x}^\mu}\, d\tau$

(c) Conserved Momenta $p^\mu = m_0 \dot{x}^\mu = (\gamma m_0 c, \gamma m_0 \vec{v})$
Energy Momentum

Lorentz Factor $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

3. Spin Mechanics

disagreement over "right" Kinematic equations & Lagrangian. Change in Spin Contributes to the momenta & energy

$p^\mu = m_0 \dot{x}^\mu + \dot{S}^{\mu\alpha} \dot{x}_\alpha$

$S^{\mu\nu} = {}^*(p \wedge S)^{\mu\nu}$,

$\dot{S}_{\mu\nu} = \dot{x}_\mu p_\nu - \dot{x}_\nu p_\mu$

$p_\mu S^\mu = 0$ Frenkel/Dixon rule

S^μ = Pauli-Lubanski spin Polarization

3.2

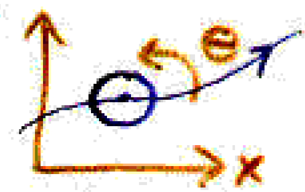
B The Clifford Manifold

(Apologies to J.S.R. Chisholm)

1. The world is a Polydimensional Continuum

(a) Generalized Coordinates $\Sigma = \Sigma(q^A)$ for each basis multivector $E_A : \{1, e_1, e_2, \cdots, e_1 \wedge e_2 \cdots\}$

(b) Example in 2D



$$d\Sigma = c\,dt\,E_0 + dx\,e_1 + dy\,e_2 + R\,d\theta\; e_1 \wedge e_2$$

Radius of gyration: R (~ Compton wavelength)

(c) Algebra $R(2) = \mathrm{End}\, R^{2,0}$ $\quad e_1^2 = e_2^2 = +1$

2. Invariant Length

$$d\bar{\Sigma} = c\,dt\,1 - dx\,e_1 - dy\,e_2 - R\,d\theta\; e_1 \wedge e_2$$

$$|d\Sigma|^2 = d\bar{\Sigma}\, d\Sigma = c^2 dt^2 - dx^2 - dy^2 + R^2 d\theta^2$$

OR $= d\lambda^2 = c^2 d\tau^2 + R^2 d\theta^2$

This is invariant under $O(2,2;R)$ Correlated Automorphism?

3. Spin contribution to motion

(a) The affine parameter λ is not the same as proper time τ when "spinning"

(b) Spin Factor $\quad \Gamma = \dfrac{d\tau}{d\lambda} = \dfrac{1}{\sqrt{1 - R^2\dot{\theta}^2/c^2}} = \dfrac{1}{\sqrt{1 + R^2\omega^2/c^2}}$

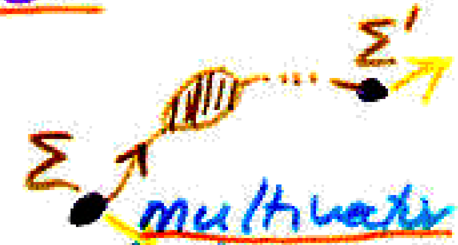
(c) Angular Velocity $\dot{\theta} = \dfrac{d\theta}{d\lambda}$ is limited by maximum "rim speed" of speed of light $R\dot{\theta} < c$ <u>But</u> the angular velocity relative to center-of-spin proper time $\omega = \dfrac{d\theta}{d\tau} = \dot{\theta}/\Gamma$ is unbounded!

3.3.6

C Polydimensional Mechanics

Continue with 2D Example

1. Action Principle



(a) Particles take minimum path in the Clifford manifold

$$A = \int m_0 c \, d\lambda = \int m_0 c \sqrt{d\bar{\Sigma} d\Sigma} \quad \text{Action Integral}$$

(b) Reparameterize in terms of "old" proper time, get spin renormalized mass

$$A = \int m_0 c \frac{d\lambda}{d\tau} d\tau = \int m_0 c \sqrt{1 + \frac{R^2\omega^2}{c^2}} \, d\tau$$

(c) Disagrees with "standard": $m \simeq \dfrac{m_0}{\sqrt{1 - R^2\omega^2/c^2}}$

2. Conserved Quantities of Motion

momentum $p = mV$

energy $E = mc^2$

Spin mass $m = m_0 \sqrt{\dfrac{1 + R^2\omega^2/c^2}{1 - V^2/c^2}}$

3 Spin Motion

(a) Standard Spin Angular Momentum goes to ∞ as rim speed approaches speed of light $\dot{\theta} = c/R$

$$L \cong mR^2\omega = \frac{m_0 R^2 \omega}{\sqrt{1 - R^2\omega^2/c^2}}$$

(b) Now we have ω unbounded, but as $\omega \to \infty$ the angular momenta tops out at $L = m_0 R c$

$$L = \frac{m_0 R^2 \omega}{\sqrt{1 + R^2\omega^2/c^2}}$$

(c) For elementary particles its natural to equate to $L \to \hbar/2$, then diameter $2R$ will approach Compton wavelength.

7
4/11

IV. Extensions of General Relativity

A. Review of standard theory

1. Affine Connected Space

Basis vectors are position dependent

(a) Vierbeins $\quad e_\mu(x^\alpha) = \partial_\mu \Sigma = h_\mu{}^\nu \hat{e}_\nu$

(b) Connection $\quad \partial_\alpha e_\mu = \Gamma_{\alpha\mu}{}^\beta e_\beta$

(c) Metric $\quad g_{\alpha\beta} = e_\alpha \cdot e_\beta$

$$|d\Sigma|^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

In Riemann Space Liebniz Rule holds:

$$\partial_\mu g_{\alpha\beta} = (\partial_\mu e_\alpha)\cdot e_\beta + e_\alpha \cdot (\partial_\mu e_\beta)$$

can solve for Christoffel Symbol $\quad \Gamma_{\mu\alpha}{}^\delta = \frac{1}{2} g^{\delta\mu}(\partial_\mu g_{\alpha\beta} - \partial_\alpha g_{\mu\beta} - \partial_\beta g_{\mu\alpha})$

2. Geometrodynamics

(a) Action Principle still holds,

$$A = \int m_0 c \sqrt{dx^\alpha dx^\beta g_{\alpha\beta}} \quad \text{minimize path length}$$

(b) Geodesics are "shortest paths in curved space

linear Motion $\quad \dot{p}^\mu = -p^\alpha \dot{x}^\beta \Gamma_{\alpha\beta}{}^\delta$

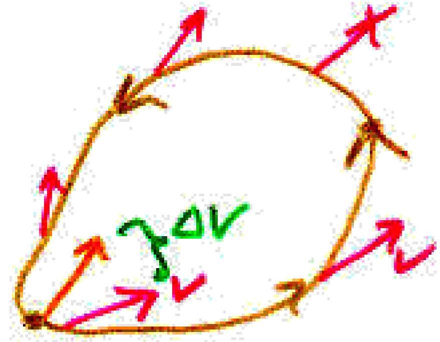
spin $\quad \dot{S}^\mu = -S^\alpha \dot{x}^\beta \Gamma_{\alpha\beta}{}^\delta$

(c) Weak equivalence principle: all particles follow same geodesic path independent of internal structure (i.e. mass and spin)

③ Curvature

Einstein Equivalence Principle

Curved space is equivalent to gravitational forces.



(a) Riemann Space

Parallel transport of vector around closed loop (area $\Delta A^{\alpha\beta}$) has length unchanged but rotated

$$\Delta V^{\nu} = R_{\alpha\beta\mu}{}^{\nu}\, V^{\mu}\, \Delta A^{\alpha\beta}$$

CURVATURE TENSOR: $R_{\alpha\beta\mu\nu} = e_{\beta}\cdot[\partial_{\mu},\partial_{\nu}]\,e_{\alpha}$

(b) Weyl Space: scale is function of position

$$\partial_{\mu} g_{\alpha\beta} = (\partial_{\mu} e_{\alpha})\cdot e_{\beta} + e_{\alpha}\cdot(\partial_{\mu} e_{\beta}) + \phi_{\mu}\, g_{\alpha\beta}$$

under parallel transport, length can change

$$\Delta V^{2} = V^{2}\,(\partial_{\mu}\phi_{\nu} - \partial_{\nu}\phi_{\mu})\,\Delta A^{\mu\nu}$$

(c) Spinning Particles

Papapetrou (1951) argues spin couples directly to curvature

$$\dot{p}^{\mu} = -p^{\alpha}\,\dot{x}^{\beta}\,\Gamma_{\alpha\beta}{}^{\mu} - \tfrac{1}{2} R_{\epsilon\sigma}{}^{\mu\nu}\,\dot{x}_{\mu}\, S^{\epsilon\sigma}$$

But this violates Einstein's weak equivalence principle, a spinning particle will take a different path. This is an open question which is currently debated. No agreed form of kinematic equation, nor proper Lagrangian to use, can't get from "minimum path" idea.

9
4.3

B Polyaffine Space

Postulate: at each point in space can always choose an orthonormal fiducial Clifford group (where $e_\mu e_\nu + e_\nu e_\mu = \pm 2\delta_{\mu\nu}$).

1. Generalized Algebra

(a) Local algebra automorphism invariance: can reshuffle geometry at each point. No preferred direction for "vector". The "dimension" (grade of blade, rank) of multivector basis element depends on observer.

(b) Basis Multivectors are: position dependent. Relate to fiducial set with Geobeins (geometry legs) $E_A(q^D) = \Delta_A{}^B \hat{E}_B$

(c) Generally is a Jordan Algebra. $\frac{1}{2}\{E_\alpha, E_\beta\} = G_{\alpha\beta}{}^C E_C$

Product of two basis vectors $e_1 e_2$ might not yield the "bivector" $E_{(12)} = e_1 \wedge e_2$

I will restrict to following by fiat

- Commuting Scalar $[E_A, E_0] = 0$, $A \neq 0$
- Weyl Space $E_0 E_0 = g_{00} E_0$, $g_{00} \neq 1$
- Clifford Product $e_\alpha e_\beta = g_{\alpha\beta} E_0 + e_\alpha \wedge e_\beta$

4.4 10

2. Poly-Connection Coefficients

The definition is trivial $\frac{\partial E_A}{\partial q^B} = \Lambda_{AB}{}^{C} E_C$ but $\Lambda_{AB}{}^{C}$ are not all independent. Their form will depend up the restrictions on the algebra rule set which tells the type of space (e.g. Riemann, Weyl, and we need new names).

(a) Return to 2D example $\{E_0, e_1, e_2, e_1 \wedge e_2\}$

$$\partial_A e_\mu \equiv \sigma_{A\mu} E_0 + \Gamma_{A\mu}{}^{\nu} e_\nu + \lambda_{A\mu}\, e_1 \wedge e_2$$
$$\partial_A E_0 \equiv -\phi_A E_0 + M_A{}^{\mu} e_\mu + N_A\, e_1 \wedge e_2$$

(b) By algebra rule $E_3 = e_1 \wedge e_2 = [e_1, e_2]\frac{1}{2}$ [Not generally true!] the connection on E_3 is completely determined by above. Note however the Liebnitz Rule does NOT hold for dot or wedge product (ok for direct)

$$\partial_A e_1 \wedge e_2 \neq (\partial_A e_1) \wedge e_2 + e_1 \wedge (\partial_A e_2)$$
$$= \tfrac{1}{2}[\partial_A e_1, e_2] + \tfrac{1}{2}[e_1, \partial_A e_2]$$
$$= \Gamma_{A\alpha}{}^{\alpha}\, e_1 \wedge e_2 \qquad \text{Standard term}$$
$$+ (\lambda_{A1} g_{22} - \lambda_{A2} g_{12}) g_{00}\, e_1$$
$$+ (\lambda_{A2} g_{11} - \lambda_{A1} g_{12}) g_{00}\, e_2$$

No part proportional to E_0!

11

4.5

(c) Relate to metric

$$\partial_A(g_{\alpha B}E_0) = (\partial_A g_{\alpha B})E_0 + g_{\alpha B}(\partial_A E_0)$$

$$\frac{1}{2}\partial_A\{e_\alpha, e_\beta\} = \frac{1}{2}\{\partial_A e_\alpha, e_\beta\} + \frac{1}{2}\{e_\alpha, \partial_A e_\beta\}$$

Get relations between various connections $M_A{}^{\mu} g_{\mu\delta} = \sigma_{A\delta}\, g_{00}$

Weyl Connection $\partial_\mu g_{\alpha\beta} = \Gamma_{\mu\alpha}{}^{\delta} g_{\delta\beta} + \Gamma_{\mu\beta}{}^{\delta} g_{\delta\alpha} + \underbrace{\phi_\mu g_{\alpha\beta}}$

---

③ Generalized Curvature (Comments)

(a) Definition Easy $[\partial_A, \partial_B] E_C = \mathcal{Z}_{ABC}{}^{D} E_D$
in our 2D case the form would be
$$[\partial_\mu, \partial_\nu] e_\alpha = R_{\mu\nu\alpha}{}^{\beta} e_\beta + W_{\mu\nu\alpha} E_0 + V_{\mu\nu\alpha}\, e_1 e_2$$

(b) Parallel transport of vector about closed loop may be rotated into a bivector or scalar!
Paths can be part line, part area, part scalar (??)

(c) Polydimensional Equivalence Principle
this type of curvature is equivalent to forces which couple spin and linear motion (describe other forces than just gravity).

12

4/16

C. Poly-Geometrodynamics

— Brief notes —

① Polygeodesics

Particles will follow shortest multivector path in curved space. Form is analogous to geodesics, except will full poly connection,

$$\ddot{q}^A + \dot{q}^B \dot{q}^C \Lambda_{BC}^{\ \ A} = 0$$

Note the dot is derivative with respect to affine length, not same as proper time!

② Our 2D example

Simplified case where $E_0 \equiv 1$

$$0 = \ddot{x}^1 + \dot{x}^\mu \dot{x}^\nu \Gamma_{\mu\nu}^{\ \ 1} + R\dot{\theta}\dot{x}^\alpha (g_{22}\lambda_{\alpha 1} - g_{21}\lambda_{\alpha 2})$$

$$0 = \ddot{x}^2 + \dot{x}^\mu \dot{x}^\nu \Gamma_{\mu\nu}^{\ \ 2} + \underbrace{R\dot{\theta}\dot{x}^\alpha (g_{11}\lambda_{\alpha 2} - g_{21}\lambda_{\alpha 1})}_{\text{Spin Contributes to linear motion (momentum)}}$$

$$0 = R\ddot{\theta} + R\dot{\theta}\dot{x}^\mu \Gamma_{\mu\nu}^{\ \ \nu} + \underbrace{\dot{x}^\mu \dot{x}^\nu \lambda_{\mu\nu}}$$

these are not "standard", but reasonable.

③ Spin as a source

Mass is source of standard Riemann Curvature, here "Spin" will also be a source. Its not the same as torsion.

V Epagogy

3

Principle of Relative Dimension. *In standard relativity, a scalar (point) is the same to all observers, in all coordinate systems. While a line may be bent due to curvature, its length is unchanged. Now Dimension is in the eye of the beholder. The geometric rank that an observer assigns to an object (e.g. bivector) is a function of the observer's frame. It might be possible to logically extend this statement to say that there is no absolute dimension to the universe (i.e. you can't say space is 4D or 5D).*

Polydimensional Isotropy. *'No preferred direction' is extended to mean that there is no absolute direction to which you can assign the geometry of a vector. For example, if we turn out the lights and exchange the basis vectors for their dual trivectors in all formulas in 4D, you can't tell that a change was made.*

The Greider Maxima. *To be complete, the laws of physics must be multivectorial in form (having scalar, vector, bivector etc. parts). Every geometric piece of a multivector equation must be physically interpretable. A separate 'Spin space' is an unneeded construct.*

Polydimensional Covariance. *The laws of physics should be form invariant under local automorphism transformations, which reshuffle the physical geometry. Spin gauge theory (in spinor space) is not therefore an artifact of spin space, it is a manisfestation of this broader classical principle.*