Should Metric Signature Matter in Clifford Algebra Formulations of Physical Theories?

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Abstract

Standard formulation is unable to distinguish between the (++++-) and (-----) spacetime metric signatures. However, the Clifford algebras associated with each are inequivalent, \( \mathbb{R}(4) \) in the first case (real 4 by 4 matrices), \( \mathbb{H}(2) \) in the latter (quaternionic 2 by 2). Multivector reformulations of Dirac theory by various authors look quite inequivalent pending the algebra assumed. It is not clear if this is mere artifact, or if there is a right/wrong choice as to which one describes reality. However, recently it has been shown that one can map from one signature to the other using a "tilt transformation" [see P. Lounesto, "Clifford Algebras and Hestenes Spinors", Found. Phys. 23, 1203-1237 (1993)]. The broader question is that if the universe is signature blind, then perhaps a complete theory should be manifestly tilt covariant. A generalized multivector wave equation is proposed which is fully signature invariant in form, because it includes all the components of the algebra in the wavefunction (instead of restricting it to half) as well as all the possibilities for interaction terms.

Note: this talk was based on the preprint of the same title, 12 pages at gr-qc/9704048.
Index to Transparencies Color and BW.

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- index of other talks at: http://www.clifford.org/~wpezzag/talks.html
- This URL: http://www.clifford.org/wpezzag/talk/97ams/
  Updated: 2005May20
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and

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- Mathematics: Clifford Algebra of (----+) inequivalent to (++++)
- Physics: Principles are signature blind (or not?)
- Question: How Reconcile Math & Physics?

http://xxx.lanl.gov/abs/gr-qc/9704048

Presentation at the 1997 Spring Western Sectional Meeting of the American Mathematical Society, Corvallis, OR, April 19-20, 1997, Special Section on Octonions and Clifford Algebras.
# Two Possible Metric Signatures of 4D Spacetime

## Signature

<table>
<thead>
<tr>
<th></th>
<th>'East'</th>
<th>'West Coast'</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma_3$</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma_4$</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Vector</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma_1\gamma_2$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma_2\gamma_3$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma_3\gamma_1$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma_4\gamma_1$</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\gamma_4\gamma_2$</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Bivector</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma_4\gamma_3$</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Trivector</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Pseudovctor)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma_4\gamma_2\gamma_3$</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\gamma_4\gamma_3\gamma_1$</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$\gamma_1\gamma_2\gamma_3$</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Pseudoscalar</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma_1\gamma_2\gamma_3\gamma_4$</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

### Count

|        | 6 | 10 | 6 | 10 |

### Lorentz Group

- same for both!
- SL(2,C)
- (Duals to vectors)
- does not commute!

### Inequivalent

- Matrix Rep: $R(4)$ | $H(2)$
**Matrix Representation vs. Metric**

<table>
<thead>
<tr>
<th>NUMBER OF POSITIVE METRIC DIMENSIONS</th>
<th>NUMBER OF NEGATIVE METRIC DIMENSIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>R</td>
</tr>
<tr>
<td>1</td>
<td>$^2$R</td>
</tr>
<tr>
<td>2</td>
<td>R(2)</td>
</tr>
<tr>
<td>3</td>
<td>C(2)</td>
</tr>
<tr>
<td>6</td>
<td>H(4)</td>
</tr>
<tr>
<td>7</td>
<td>C(8)</td>
</tr>
</tbody>
</table>

**Key**

\[ X(n) = \text{n by n matrix, with the components being:} \]

- Real Numbers \( X = "R" \)
- Complex Numbers \( X = "C" \)
- Quaternionic Numbers \( X = "H" \)

\[ 2X(n) = \text{(Two) Block Diagonal Matrix} \]

\[
\begin{pmatrix}
X(n) & 0 \\
0 & X(n)
\end{pmatrix}
\]
Metaprinicples of Physical Symmetry

All the laws of physics must obey:

1. Isotropy of 3D
   - Pythagorean: No special direction $\sqrt{x^2 + y^2 + z^2}$ for $(+++)$ or $(---)$
   - Map: Invariance under $SO(3)$ Rotations
   - Law: Conservation of Angular Momentum

2. Parity Invariance
   - Map: Left to right handed coord system
   - Law: Conservation of Parity
   - Neutrinos/Weak Interactions do NOT obey! Expt/Formulation can tell if left or right handed!

3. Principles of Relativity
   - Map: Invariance under $SL(2,\mathbb{C})$ Lorentz Group
   - Law: Can't exceed speed of light $\pm d\tau^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$ (+++ and ----)
   - Covariance: Form invariance under local changes of coordinate systems (gravity)
1. Transformation \((+++) \rightarrow (---)\)

\[ \hat{e}_j \cdot \hat{e}_k = g_{jk} \rightarrow -g_{jk} \]
\[ dx^j \rightarrow dx^j \]
\[ \hat{e}_j g^{jk} \hat{e}_k = \nabla \rightarrow -\nabla \]

2. Gibb's Cross Product

- Is NOT signature covariant
  \[ \nabla \times \vec{E} = -\hat{e}_c \vec{B} \leftrightarrow -\nabla \times \vec{E} = -\hat{e}_c \vec{B} \]
- Is Faraday's law able to distinguish if space is \((+++)\) or \((---)\)?
- Or is Gibb's Algebra Flawed? (yes)

3. Clifford Product

- gives signature invariant formulation!
  \[ \nabla \wedge \vec{E} = -\hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_c \vec{B} \]
  \[ \nabla \cdot \vec{E} = \varepsilon \]

\[ \varepsilon^{i_1 i_2 i_3 i_4} = -1 \text{ vs } +1 \text{ same in } (+++) \text{ or } (---) \]
"Non-Issue" in standard Physics

Many Papers!

Local Signature Changes (function g position)
Quantum Mechanics is NOT Signature Form Invariant

Can QM distinguish between (++++) and (----)?

1. Definitions of Anti-Involutions change

\[ j^\mu = \psi + \gamma^\mu \psi \mu \psi \]

Form invariant

\[ (++++) \left\{ \begin{array}{l}
\psi^\dagger = \overline{\psi} i \gamma^0 \\
\overline{\psi}^\mu = -\gamma^\mu \overline{\psi} \\
\overline{\psi}^\dagger = +\psi^\dagger
\end{array} \right\} (----) \]

2. Observables (quadratic forms)

Can be made invariant if \( \psi \) changes

\[ p^\mu = -\frac{i}{2} g^\mu\nu \left[ \psi^\dagger (\partial_\nu \psi) - (\partial_\nu \psi^\dagger) \psi \right] \]

\[ g^\mu\nu \rightarrow -g^\mu\nu \]

\[ \psi \rightarrow i\gamma^2 \psi^* \quad \text{Non Trivial!} \]

3. Weak Signature Covariant Principle

- QM displays Weak (not strong) Signature Covariance
- Observables obey Strong Signature Covariance
- But Wavefunctions must transform non-trivially
(Non-Standard) Multivector Quantum Mechanics

- $\gamma^\mu =$ basis vectors of space-time
- Real 4D Clifford Algebra - no "i"
- No spinors or spin space, $\Psi$ a multivector

<table>
<thead>
<tr>
<th>Signature:</th>
<th>$(+++--)$</th>
<th>$(--+-+)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Klein-Gordon Equation</td>
<td>$(\Box^2-m^2)\phi=0$</td>
<td>$(\Box^2+m^2)\phi=0$</td>
</tr>
<tr>
<td>Factorization</td>
<td>$(\Box-m)(\Box+m)$</td>
<td>$(\Box-im)(\Box+im)$</td>
</tr>
<tr>
<td>Multivector Potential</td>
<td>$\Psi=(\Box+m)\phi$</td>
<td>no $i$!</td>
</tr>
<tr>
<td>&quot;Dirac&quot; Eqn.</td>
<td>$\Box\Psi=m\Psi$</td>
<td>&quot;Wrong&quot;</td>
</tr>
<tr>
<td>Signature Invariant Forms Multivector Dirac (Pezzaglia)</td>
<td>$\Psi=\Box\phi+m\phi^\Lambda$</td>
<td>same</td>
</tr>
<tr>
<td></td>
<td>$\Box\Psi=m\Psi^\Lambda$</td>
<td>same</td>
</tr>
<tr>
<td></td>
<td>$\Gamma^2=+1$</td>
<td>$\Gamma^2=-1$</td>
</tr>
<tr>
<td>Lounesto - Hestenes</td>
<td>$\hat{\Gamma}=\gamma_{412}$</td>
<td>same</td>
</tr>
</tbody>
</table>
**Signature Invariance and the Interface of Math/Physics**

<table>
<thead>
<tr>
<th>Mathematical Formulation</th>
<th>Physical Phenomena</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Invariant</strong></td>
<td></td>
</tr>
<tr>
<td>• Strong Signature Covariance</td>
<td></td>
</tr>
<tr>
<td>• Conserved Quantity?</td>
<td></td>
</tr>
<tr>
<td><strong>NOT</strong></td>
<td></td>
</tr>
<tr>
<td>• Weak Signature Covariance</td>
<td></td>
</tr>
<tr>
<td>• Flaw in Language or Artifact</td>
<td></td>
</tr>
<tr>
<td>• Internal Symmetry? (Gauge field?)</td>
<td></td>
</tr>
</tbody>
</table>
The Tilt Transformation

How map Clifford Algebra to its Complement

1. Standard Trick \((+++\to-\cdots+\to)\)
\[\gamma_u \to i\gamma_u\]

is Flawed, no "i" in Real in \(H(2)\) or \(R(4)\)

2. Tilt Transformation

- Map: \(\text{End } R^{p,2} \to \text{End } R^{2,p}\)
- \(AB \to B_e A_e + B_o A_e + B_e A_o - B_o A_o\)

3. Quadratic Forms Problematic

\((\Psi \Psi)_0 \to (\Psi \Psi)_0 \neq (\Psi \Psi)_0\)
\((\Psi \Psi)_e \to (\Psi \Psi)_e \neq (\Psi \Psi)_e\)

\[\overline{\gamma_u} = -\gamma_u\] Dirac Bar
\[\overline{\gamma_u} = +\gamma_u\] Reversion
Multivector Dirac Equation under Tilt

(Real 4D Clifford Algebra, no \(i\), no spinors)

1. Signature Invariant Form
   - \( \Box \Psi = m \Psi \Gamma + A^u \gamma_u \gamma^u \Psi \Lambda \)
   - \([\Lambda, \Gamma] = 0\), \(\Lambda^2 = -1\), \(\Gamma^2 = \pm 1\) (Signature)
   - Hestenes/Lounesto: \(\Lambda = \gamma_{12}\), \(\Gamma = \gamma_{42}\)

2. Strong Tilt Covariance
   - Does NOT follow, \(\Psi \rightarrow \tilde{\Psi}\)
   - Recover if restrict \(\tilde{\Psi} = +\Psi\) or \(\tilde{\Psi} = -\Psi\)
   - Neither properly describes Dirac Physics!

3. Weak Tilt Covariance
   - Only observables (quadratic form) invariant
   - Must Restrict \(\Psi\) to \(\Psi_e\) (even) or \(\Psi_o\) (odd)
   - Either sufficient for Dirac (8 degrees freedom)
Maximal Theory with more Phenomena

1. Use all 16 degrees of freedom
   
   \[ \psi = \sum_{A=1}^{16} \psi^A \hat{E}_A \quad \hat{E}_A \in \{ E, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\rho} \} \]

   - Left ideal structure is SPIN
   - Right ideal structure is ISO/SPIN


2. Tilt Complement may not be same form
   
   \[ \Box \psi = m \psi \quad \rightarrow \quad \Box \bar{\psi} = m \gamma_{1234} \bar{\psi} \gamma_{1234} \]

   - Appears to represent same physics! (weak signature covariance)

   \[ \Box^2 \psi = +m^2 \psi \quad \rightarrow \quad \Box^2 \bar{\psi} = -m^2 \bar{\psi} \]

3. Observables under tilt:
   
   - \( \psi \leftrightarrow \bar{\psi} \) interchanges spin \( \leftrightarrow \) isospin
   - \[ Q_{ij} = \text{Tr} (\psi \hat{E}_i \psi \hat{E}_j) = \text{Tr} (\hat{E}_i \psi \hat{E}_j \psi) \]

   - Tilt of \( Q_{ij} \) \( \rightarrow \) ? non-trivial ??
1. Explicit Form of 2-sided Couplings

\[ \nabla \Psi = \Psi (m_1 + \epsilon \gamma_\mu a_\mu^u + \hat{\epsilon} \chi) \\
+ \hat{\epsilon} \Psi (\gamma^r \gamma_\mu \Gamma^u_r + \hat{\epsilon} m_2) \\
+ \gamma^u \Psi (\gamma^r b_{\mu r} + \gamma^u A^a_a + \hat{\epsilon} \gamma^u \Lambda^a) \\
+ \epsilon \gamma^u \Psi (\gamma^r + \epsilon \gamma^u \Lambda^a + \hat{\epsilon} \phi^u) \\
+ \gamma^{\alpha \beta} \Psi (\gamma^u s_{\alpha \beta} + \gamma^{\mu \nu} R_{\alpha \beta}^{\mu \nu}) \\
\]

where \( \hat{\epsilon} = \gamma_{1234} \) and some combinations suppressed.

2. Interpretation of Phenomena

- U(1) Electromagnetism \( A^u = b_{u}^A \), \( A^u \Psi \Psi^* \)
- SU(2) Weak \( W_{\mu}^{A} = c_{u}^{A} \)
- In particular Z boson (\( W_{\mu}^{3} \)) is \( Z_{\mu} \gamma_{\mu} \gamma_{5} \delta_{12} \) which Hestenes/Lounesto mistake for EJM \( A_{\mu} \)!

3. Under Tilt transformation, \( \Psi \leftrightarrow \bar{\Psi} \)

- \( m_1 \leftrightarrow -m_2 \)
- \( \gamma^u \gamma_\mu \rightarrow -\gamma^u \gamma_\mu \)
- \( \Gamma_{\alpha \beta \gamma} \leftrightarrow -\epsilon^{\alpha \beta \gamma \delta} \epsilon_{\delta} \Gamma_{\alpha \beta \gamma} \)
- Rest invariant!
Summary: "Clarify Questions"

1. Mathematics: Does the existence of the Tilt Complement negate any arguments based on Special Structure about Right/Wrong Signature? [N.b. Salingaros, Greider, Hestenes ...]

2. Physics: If metaprinciple of signature invariance is true, then must all formulations be (strong/weak) tilt Covariant? [Necessary?, Sufficient? Both?]

3. The Interface: If tilt complement describes different phenomena, then should we complete the symmetry by expanding to formulation? [Covarying Theory]

Maximal Model

Tilt Covariant

Formula #1 (+-+-)

Sector #1 Physics (Spin)

Formula #2 (-++-)

Sector #2 Physics (Isospin)