Why does "real" physics need "imaginary" numbers?

A history of physical applications of geometric algebras?

Presenter: William M. Pezzaglia Jr. (wpezzag@clifford.org)

Department of Physics, Santa Clara University, Santa Clara, CA 95053, USA

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Abstract

Certain mathematical systems: complex algebra, Gibbs vectors (1881) and Sylvester's matrices (1850), have wide applications in physics, engineering and chemistry. However there were many siblings (Hamilton's Quaternions 1843, Grassmann's Algebra of Extension 1844, Cayley's Octonions 1845) that fell into obscure disuse but have recently found new applications which we will review. In particular, William Kingdon Clifford proposed (1876) a universal geometric algebra which combined features of all the above. Due to his untimely death this extraordinary system, in which one can add a vector to a scalar to a plane, was ignored for nearly 100 years. Recent revival (last 20 years) has shown that undergraduates can more quickly learn and apply Clifford vectors than Gibbs vectors; becoming quite excited with the interpretation of "i" as the volume of 3 space, seeing "planes" as things that cause rotations (or Lorentz transformations), and being able to do divergence, curl and gradient in a single equation. New extensions of a generalized geometric calculus hold promise for completely new approaches to unified physical theories.

This will be a very general talk, of interest to undergraduates (I myself was introduced to the subject as a sophomore, and was captivated by the insights it gave that I found nowhere else).

- Index of other talks at: http://www.clifford.org/wpezzag/talks.html
- This URL: **http://www.clifford.org/wpezzag/talk/1998math/98math.pdf**
- Updated: 2007Dec26

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Why does “real physics” need “imaginary” numbers?

A history of physical applications of geometric algebras.

William M. Pezzaglia Jr.

Department of Physics
Santa Clara University
Santa Clara, CA 95053
mailto:wpezzag@clifford.org, http://www.clifford.org/~wpezzag

- **Shadow of Shadows:** History of imaginaries
- **Geometric Algebra:** Adding a vector to a plane
- **Physical Applications:** New math => New physics?

1998 Jan 20, Wed 4-5 pm, 207 O'Connor, Santa Clara Univ Math Dept Colloq.
http://www.clifford.org/~wpezzag/talks.html#98jan20

II. Imaginary Algebras

A. Complex Algebra

- 1. Complex plane as 2D Space
- 2. Differentiation (Cauchy-Riemann)
- 3. Cauchy Integral Theorems

B. Hamilton's Quaternions (1843)

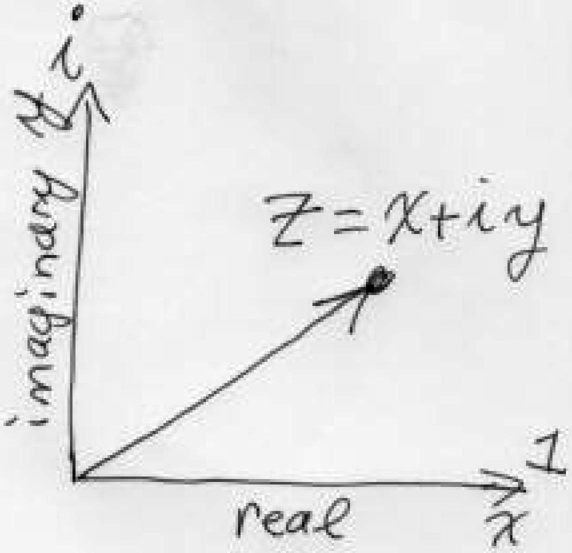
- 1. Algebra of Scalar + Vector
- 2. Rodrigues & Cayley Rotations
- 3. Quaternionic Analysis

C. Other Algebras

- 1. Cayley Octonions, 7D cross product (1881)
- 2. Abstract Algebra Properties
- 3. Historical Overview

II.A. Complex Algebra

- **1. The Complex Plane**
  - (a) Imaginary: $i^2=-1$
    1545 Cardan “Useless Roots
    1637 Descartes “imaginary”
    1831 Gauss: complex number
  - (b) 2D Space
    1673 John Wallis
    1793 Casper Wessel
    1821 Cauchy “complex plane”
  - (c) Rotations in 2D
    1743 Euler: $e^{i\phi}=\cos\phi+\mathbf{i}\sin\phi$
    $\boxed{\text{1806 Argand: } z' = e^{i\phi} z}$



- 2. Analysis
  - (a) Complex Functions $f(z) = g(z) + i\,h(z)$
    1747 d’Alembert
  - (b) Derivatives for $dz^n = nz^{n-1}dz$ need

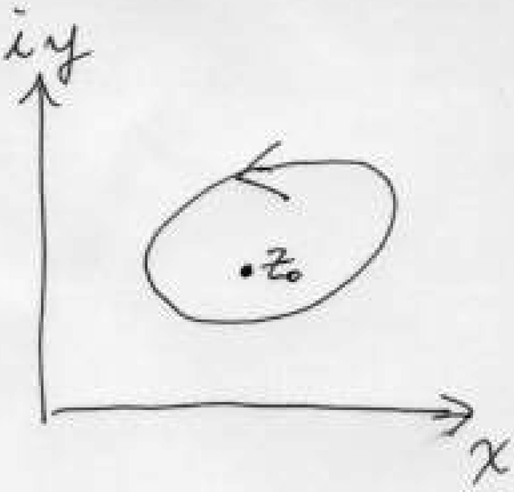
$$\boxed{\text{Cauchy Riemann Conditions} \quad \frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} \qquad \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}}$$

  - (c) Analytic Functions
    are Harmonic

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 0$$

- **3. Integration**

  - (a) Cauchy Integral Theorem



$$\oint f(z)\,dz = 0$$

  - (b) Cauchy Kernel $K(z, z_0) = \frac{1}{2\pi i\,(z - z_0)}$

$$\oint K(z, z_0)\, f(z)\,dz = f(z_0)$$

  - (c) Generalizations for non-analytic

I've seen several versions — they involve an integral over the area enclosed by the contour — not something you would usually do. It picks up contributions from poles & cuts.

B. Quaternions

Sir William
Rowan Hamilton
1805-1865

$i^2=j^2=k^2=ijk=-1$

Famous equation, carved in Brougham Bridge Oct 6, 1843.

1. Anticommutivity is Perpendicularity:

$$i^2 = j^2 = -1 \qquad \boxed{ij = -ji}$$

Define: $k = ij \Rightarrow j = ki = -ik$

$i = jk = -kj$

As a "Group" $\{i, j, k\}$ describe $SU(2)$

Quaternion: $H = \omega 1 + xi + yj + zk$

Adjoint $H^{+} = \omega 1 - xi - yj - zk$

Norm: $H^{+} H = \omega^2 + x^2 + y^2 + z^2$

II.A.2 Quaternionic Algebra

(a) Hamilton defines "scalar" & "Vector" $H = \omega + \vec{V}$

(b) Product of two Vectors:

$$AB = (A_1\hat{i} + A_2\hat{j})(B_1\hat{i} + B_2\hat{j})$$

$$= A_1B_1\hat{i}^2 + A_2B_2\hat{j}^2 + A_1B_2\hat{i}\hat{j} + A_2B_1\hat{j}\hat{i}$$

$$= -(A_1B_1 + A_2B_2) + (A_1B_2 - A_2B_1)\hat{k}$$

$$AB = -\underset{\text{scalar}}{A\cdot B} + \underset{\text{vector}}{A\times B}$$

$$\boxed{\begin{aligned} \mathbf{A\cdot B} &= -\tfrac{1}{2}\{\mathbf{A},\mathbf{B}\} = -\tfrac{1}{2}(AB+BA) \\ \mathbf{A\times B} &= \tfrac{1}{2}[\mathbf{A},\mathbf{B}] = \tfrac{1}{2}(AB-BA) \end{aligned}}$$

(c) Calculus: Hamilton introduces ∇ operator:

$$\vec{\nabla} \equiv \hat{i}\partial_x + \hat{j}\partial_y + \hat{k}\partial_z$$

$$\vec{\nabla}\vec{E} = -\nabla\cdot E + \nabla\times E$$

(1860) Maxwell's Equations originally written with Quaternions.

$$\nabla E = -\rho - \frac{\partial B}{\partial t} \qquad \nabla B = +\frac{\partial E}{\partial t}$$

These 2 equations encode the 4 Maxwell eqns!

II.A.3 Rotations

(a) Reflection of a vector $\vec{V} \equiv (xi + yj + zk)$

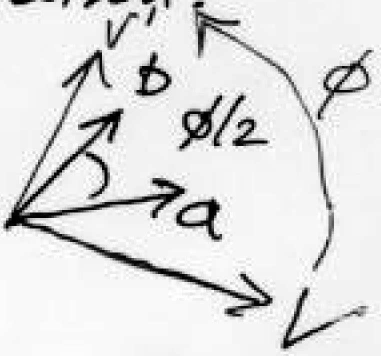
$$-i\vec{V}i = xi - yj - zk$$

$-\hat{n}\vec{V}\hat{n}$ will reflect V around $\hat{n}$

(b) Two Reflections make a rotation!

$$V' = baVab$$

will rotate vector V in plane described by $\hat{a}$ & $\hat{b}$ by angle ϕ



(c) Exponential Form

$$-ba = +a\cdot b + a\times b$$

$$= \cos\phi/2 + \hat{n}\sin\phi/2 \equiv e^{\hat{n}\phi/2}$$

where $\hat{n}^2 = -1$, $\hat{n}$ is unit vector ⊥ to plane described by vectors a & b.

$$\boxed{V' = e^{\hat{n}\phi/2}\, V\, e^{-\hat{n}\phi/2}}$$

Rotation about arbitrary axis $\hat{n}$ by angle ϕ

④ Quaternionic Analysis

(a) Functions of quaternion coordinate are really 4D (cartesian)

$$H = \omega \hat{1} + x\hat{i} + y\hat{j} + z\hat{k}$$

$$= \omega + \vec{r}$$

$$\mathcal{F}(H) = \mathcal{F}(\omega+\vec{r}) = f_0(\omega+\vec{r}) + \vec{f}(\omega+\vec{r})$$

(b) Differentiation has issues of order due to non-commutivity:

$$dH^3 = \begin{cases} 3H^2 dH & ? \\ 3dH\, H^2 & ? \\ 3H dH H & ? \end{cases}$$

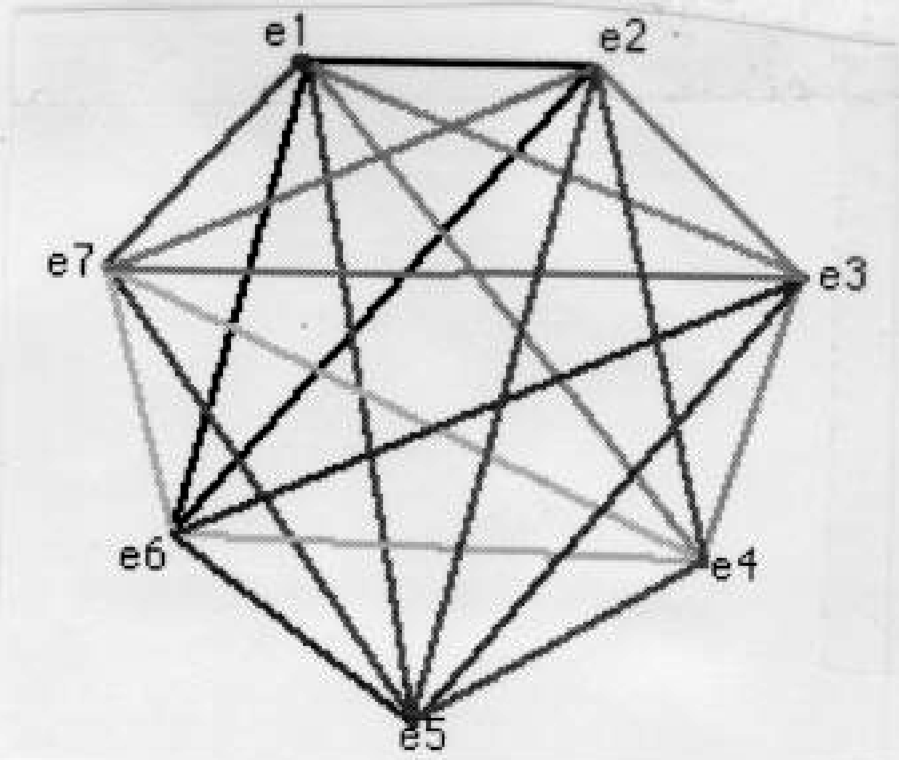
(c) Lounesto p.75 states generalizations for Cauchy-Riemann are:

$$\boxed{\begin{aligned} \frac{\partial f_0}{\partial \omega} &= \vec{\nabla}\cdot\vec{f} \\ -\frac{\partial \vec{f}}{\partial \omega} &= \nabla f_0 + \vec{\nabla}\times\vec{f} \end{aligned}}$$

what about integration? has it been done?

Cayley's Octonions (1881)

- There are 480 different ways to write an octonion multiplication table. Here is a geometric representation of the way preferred by Geoffrey Dixon:



$e_0 \equiv 1$

$(e_n)^2 = -1 \quad n \neq 0$

$e_i e_j = -e_j e_i$

-

Non-associative

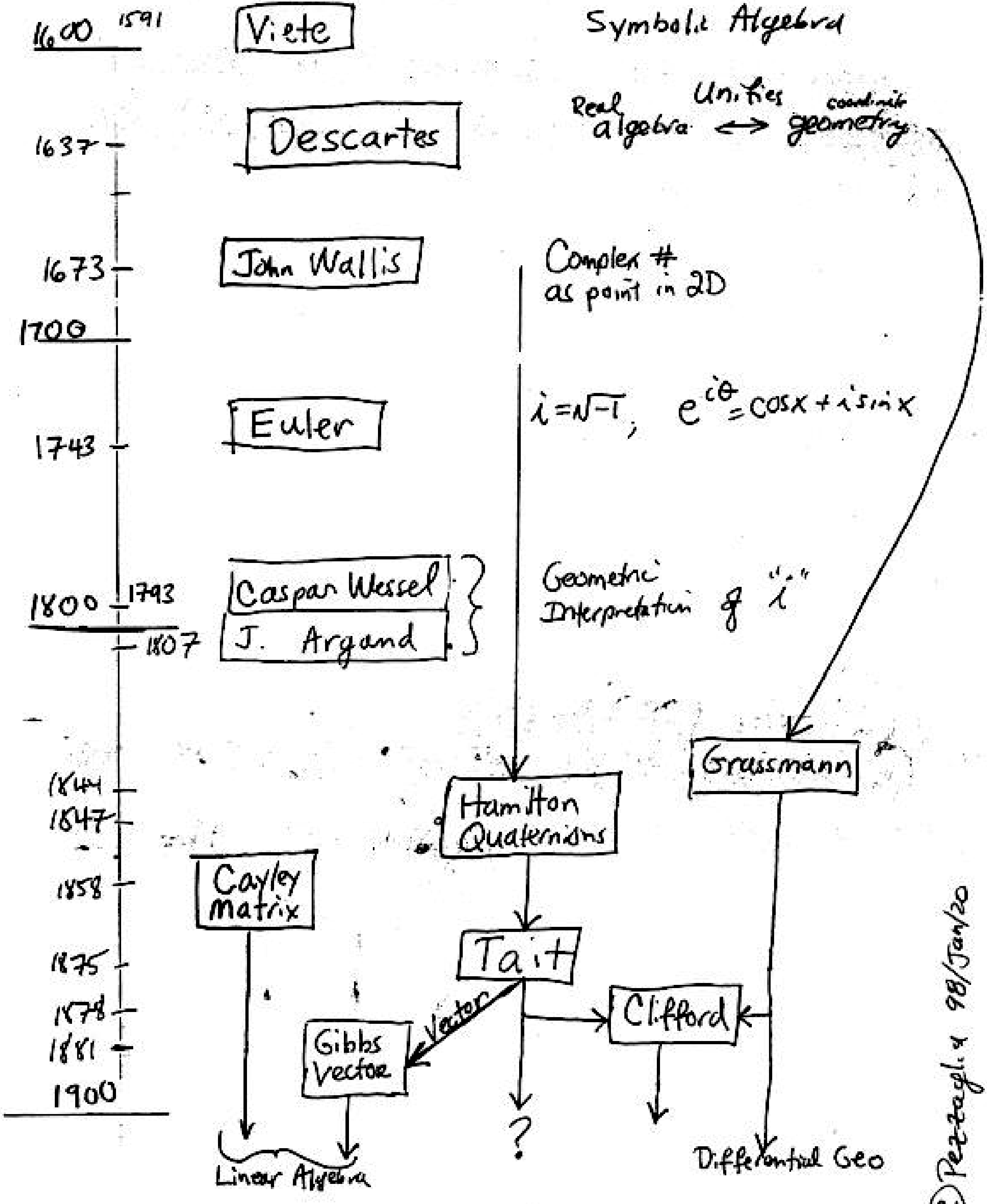
$(ab)c \neq a(bc)$

- In the heptagon of imaginary octonions {e1,e2,e3,e4,e5,e6,e7}, there are 7 triangles (6 colors and 1 black). The product of any two imaginary octonions is the third imaginary octonion in their triangle, with + sign if the product is a clockwise rotation and - sign if counterclockwise. The algebraic rule for this product is determined by e(a)e(a+1) = e(a+5). If (a+5) is greater than 7, use (a-7).

- Hurwitz' theorem (1898) that any normed division algebra over **R**, with unit element, is isomorphic to **R,C, H**(quaternions) or **O**(octonions).

II.C.1

History of Abstract Algebra



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II.C.2 Types of Algebras

	$(AB)C=A(BC)$	NON-Associative
Abelian $A \otimes B = B \otimes A$	Complex Field: C Real Field: R	Jordan Algebra $[e_i, e_j] = g_{ijk} e_k$
Non-Abelian	Matrix Algebra Cayley Dyadics **Clifford Algebra** non-isotropic: **Hamilton's Quaternions**, Pauli Matrices, Dirac Matrices isotropic: **Grassmann**, Differential Forms	Gibbs Vectors Lie Algebra $[e_i, e_j] = c_{ijk} e_k$ Cayley Octonions

Associativity

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III. Geometric Algebra

A. Gibbs Vectors (1881)

- 1. Gibbs Vector Algebra
- 2. Coordinate free & Isotropic properties
- 3. The Vector-Quaternion Debate (1891-4)

B. Grassmann's Algebra

- 1. Grassmann's Extended Quantities
- 2. Hodge Dual, the Dot Product
- 3. Differential Forms, Stokes Theorem

C. Clifford Algebra

- 1. The Clifford Group, Matrix Representation
- 2. Adding Vectors to Planes
- 3. Geometric Calculus

III.A.1

J. Willard Gibbs (1839-1903)

"America's First Theoretical Physicist"

appointed Prof. of Math Physics, Yale 1871 without pay until 1880!

> "But I do not so much desire to call your attention to the diversity of the applications of multiple algebra, as to the simplicity and unity of its principles. The more we study the subject, the more that we find all that is most useful and beautiful attaching itself to a few central principles. We begin by studying multiple algebras; we end, I think, by studying **Multiple Algebra**." - *(1885) presidential address given to the American Association for the Advancement of Science.*

- Define Dot Product Positive $\begin{cases} i \cdot i = +1 \\ j \cdot j = +1 \\ k \cdot k = +1 \end{cases}$
- Antisymmetric Cross Product $\begin{cases} k = i \times j = -j \times i \\ i = j \times k \\ j = k \times i \end{cases}$
- Algebra is Non-Associative $A \times (B \times C) \neq (A \times B) \times C$

III.A.2

Why Use Gibbs Vectors in Physics?

Example: Electromagnetism in 3D

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \left\{ \begin{array}{l} \partial_y E_z - \partial_z E_y = -\partial_t B_x \\ \partial_z E_x - \partial_x E_z = -\partial_t B_y \\ \partial_z E_y - \partial_y E_z = -\partial_t B_z \end{array} \right.$$

Coord Free — coordinate dependent!

Gibbs vectors encode the metaprinciple of Isotropy (of the Universe)

Consider <u>Complex</u> Gibbs Vectors

$$\vec{F} = \vec{E} + i\vec{B}$$

$$\nabla \cdot F = e \quad \left\{ \begin{array}{l} \nabla \cdot E = e \\ \nabla \cdot B = 0 \end{array} \right.$$

$$\nabla \times F - i\partial_t F = J \quad \left\{ \begin{array}{l} \nabla \times E - \partial_t B = 0 \\ \nabla \times B - \partial_t E = J \end{array} \right.$$

Again we get several-equations-in-one. But what is the principle? Duality?

III.A.3. Vectors versus Quaternions

Reference: A. M. Bork, American Journal of Physics, **34**, 202-211 (1966).

As his work on the *Treatise* progressed, Maxwell wrote of the quaternion as "a flaming sword", the virtue of which lay "in enabling us to see the meaning of the question and its solution" which struggling with physical problems. Elsewhere he states that `he had been striving all his life to be freed from the yoke of the Cartesian coordinates, and had found such an instrument in the Hamiltonian quaternions.'

1890 Maxwell writes (volume 2) "A most important distinction was drawn by Hamilton when he divided the quantities with which he had to do into Scalar quantities, which are completely represented by one numerical quantity, and Vectors, which require three numerical quantities to define them. The invention of the calculus of Quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance, with the invention of triple co-ordinates by Descaretes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be the greatest use in all parts of sicence."

Peter Gurthrie Tait (student of Hamilton) in preface to third edition of Hamilton's Quaternions, says of Gibbs vectors: "... a sort of hermaphrodite monster, compounded of the notations of Hamilton and Grassmann".

1892, Lord Kelvin in a letter states: "Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have tounched them in any way including Clerk Maxwell."

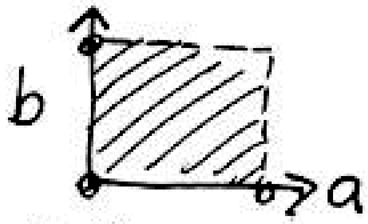
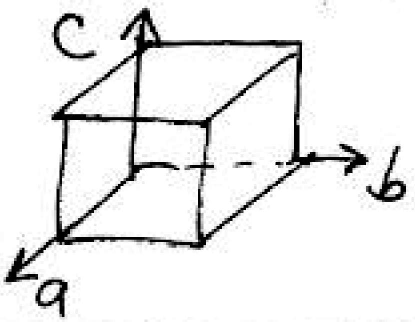
April 1893, O. Heaviside: "A vector is not a quaternion; it never was, and never will be, and its square is not negative; the supposed proofs are perfectly rotten at the core." He goes on to give Professor MacAulay, who is a "quaternionist", some advice, "A difficulty in the way is that he has got used to quaternions. I know what it is, as I was in the quaternionic slough myself once. But I made an effort, and recovered myself, and have little doubt that Prof. MacAulay can do the same."

Hermann Grassmann 1809- 1877

H Grassmann

III.B.1 Grassmann Algebra 1844

Generalized Directed Numbers for Geometric Element

Geometry	Name	Extensive	
	Point	Magnitude (scalar)	$\lvert\vec{V}\rvert$
	Line	Vector (rotor)	V
	Plane	Bivector (Leaf)	$\vec{a}\wedge\vec{b}$
	Volume	Trivector	$\vec{a}\wedge\vec{b}\wedge\vec{c}$

Exterior (outer) Product $\wedge$

Antisymmetric: $a\wedge b = -b\wedge a$

$a\wedge a = 0$

Associative $a\wedge(b\wedge c) = (a\wedge b)\wedge c$

Closed (e.g. in 3D $a\wedge b\wedge c\wedge d = 0$)

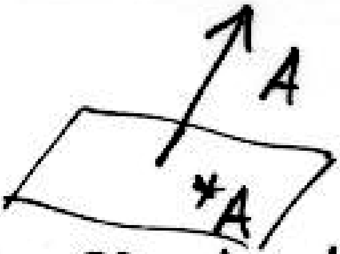
Cannot add scalar + vector!

② More on Grassmann

(a) Hodge geometric dual

$^{*}A$ = dual of A

$^{**}A = \pm A$



in 3D dual of Vector is plane

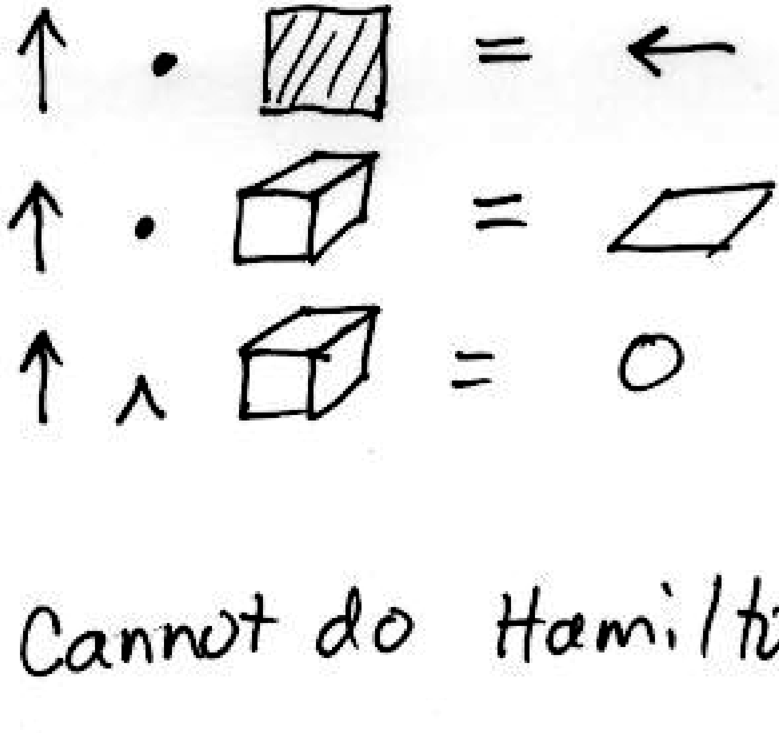
(b) Dot Product Defined

$$a \cdot b \equiv {}^{*}(a \wedge {}^{*}b)$$

(c) Gibbs Cross Product (3D)

$$a \times b \equiv {}^{*}(a \wedge b)$$

(d) Products of geometric object in 3D



No 4D object

(e) Cannot do Hamilton's Rotations!

③ Grassmann Calculus (Cartan 1923)

(a) Electromagnetism in 4D is coordinate free (unlike tensors), works in curved space!

$$\left.\begin{matrix} dF = 0 \\ {}^*d{}^*F = J \end{matrix}\right\} \quad \text{or} \quad \left\{\begin{matrix} \Box\wedge F = 0 \\ \Box\cdot F = J \end{matrix}\right.$$

(b) Differential Multiforms

$$\mathcal{D}^1 x \equiv d\vec{r} = \hat{e}_1 dx + \hat{e}_2 dy + \hat{e}_3 dz$$

$$\mathcal{D}^2 x \equiv d\underset{\sim}{a} = e_1\wedge e_2\, dxdy + e_2\wedge e_3\, dydz + \cdots$$

$$\mathcal{D}^3 x \equiv dV = e_1\wedge e_2\wedge e_3\, dxdydz$$

(c) Generalized Stokes Thm (Fundamental Theorem of Calculus)

$$\boxed{\begin{aligned} \oint \mathcal{D}^{N-1}x \cdot \mathcal{F} &= \int (\mathcal{D}^N x\cdot\nabla)\cdot\mathcal{F} \\ \oint \mathcal{D}^{N-1}x \wedge \mathcal{F} &= \int (\mathcal{D}^N x\cdot\nabla)\wedge\mathcal{F} \end{aligned}}$$

$$N=3 \quad \oint d\underset{\sim}{a}\wedge\vec{E} = \int (dv\cdot\vec{\nabla})\cdot\vec{E} = \int dv\,(\vec{\nabla}\cdot\vec{E})$$

$$N=2 \quad \oint d\vec{r}\cdot\vec{E} = \int (d\underset{\sim}{a}\cdot\nabla)\wedge E = \int d\underset{\sim}{a}\cdot(\nabla\wedge E) \equiv \int d\vec{A}\cdot(\nabla\times E)$$

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William Kingdom Clifford (1845-1879)

I hold in fact:

- (1) That small portions of space are in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them.
- (2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.
- (3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter, whether ponderable or etherial.
- (4) That in the physical world nothing else take place but this variation, subject (possibly) to the law of continuity.

"On the Space-Theory of Matter"
Proceedings of the Cambridge Philosophical Society (1876).

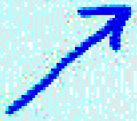
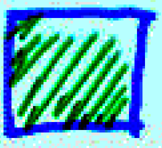
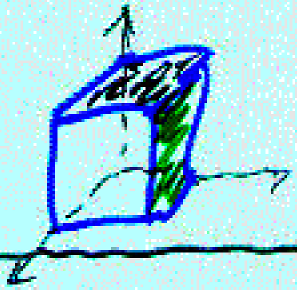
1. The Clifford Group

(a) Basis Vectors $\{\sigma_j\}$ anticommute

"Anticommutivity is perpendicularity" – Hamilton

$$\left.\begin{aligned}\sigma_1\sigma_2 &= -\sigma_2\sigma_1\\ \sigma_1\sigma_1 &= +1\end{aligned}\right\}\quad \boxed{\{\sigma_i,\sigma_j\} = 2\delta_{ij}}$$

(b) The 3D Case is isomorphic to Pauli Group

Geometry	Name	Pauli Element	Lie Group
	Scalar	1	
	Vector	$\left\{\begin{matrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{matrix}\right.$	SL(2,C)
	Bivector	$\left\{\begin{aligned}\sigma_1\sigma_2 &= i\sigma_3\\ \sigma_2\sigma_3 &= i\sigma_1\\ \sigma_3\sigma_1 &= i\sigma_2\end{aligned}\right.$	SU(2) U(2)
	Trivector	$\sigma_1\sigma_2\sigma_3 = i$	U(1)

(c) Geometric interpretation of "i"

$$i^2 = (\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_3) = -1$$

Commutes with all elements!

$$i\sigma_1 = \sigma_2\sigma_3$$

$i\sigma_1 = \sigma_2\sigma_3$

Mult by i Gives dual!

MATRIX REPRESENTATION VS. METRIC

	−	NUMBER OF NEGATIVE METRIC DIMENSIONS								
+		0	1	2	3	4	5	6	7	8
NUMBER OF POSITIVE METRIC DIMENSIONS	0	R	C	H	^{2}H	H(2)	C(4)	R(8)	^{2}R(8)	R(16)
	1	^{2}R	R(2)	C(2)	H(2)	^{2}H(2)	H(4)	C(8)	R(16)	^{2}R(16)
	2	R(2)	^{2}R(2)	R(4)	C(4)	H(4)	^{2}H(4)	H(8)	C(16)	R(32)
	3	C(2)	R(4)	^{2}R(4)	R(8)	C(8)	H(8)	^{2}H(8)	H(16)	C(32)
	4	H(2)	C(4)	R(8)	^{2}R(8)	R(16)	C(16)	H(16)	^{2}H(16)	H(32)
	5	^{2}H(2)	H(4)	C(8)	R(16)	^{2}R(16)	R(32)	C(32)	H(32)	^{2}H(32)
	6	H(4)	^{2}H(4)	H(8)	C(16)	R(32)	^{2}R(32)	R(64)	C(64)	H(64)
	7	C(8)	H(8)	^{2}H(8)	H(16)	C(32)	R(64)	^{2}R(64)	R(128)	C(128)
	8	R(16)	C(16)	H(16)	^{2}H(16)	H(32)	C(64)	R(128)	^{2}R(128)	R(256)

KEY

X(n) = n by n matrix, with the components being:

Real Numbers X = "R"
Complex Numbers X = "C"
Quaternionic Numbers X = "H"

^{2}X(n) =(Two) Block Diagonal Matrix $\left(\begin{array}{c|c} X(n) & 0 \\ \hline 0 & X(n) \end{array}\right)$

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② Properties of Clifford Algebra

(a) Associative Geometric Product $a(bc) = (ab)c$

(b) Like Hamilton's Quaternions, separate product into two parts.

$$\boxed{ab = \underset{\text{scalar}}{a\cdot b} + \underset{\text{bivector}}{a\wedge b}}$$

$$a\cdot b = \tfrac{1}{2}(ab+ba)$$

$$a\wedge b = \tfrac{1}{2}(ab-ba)$$

(c) Duals constructed by multiplying by the volume element. In 3D → "i"

$$a\times b \equiv -i\, a\wedge b$$

(d) Can do things Grassmann Can't

Grassmann	Clifford
$(e_1\wedge e_2)\cdot(e_2\wedge e_3)=0$	$(e_1e_2)(e_2e_3)$
$(e_1\wedge e_2)\wedge(e_2\wedge e_3)=0$	$=e_1e_3$

III.C 3 Geometric Calculus

ⓐ Derivative of a Vector $\vec{E}$

$$\nabla E = \underset{\text{scalar}}{\nabla\cdot E} + \underset{\text{Bivector}}{\nabla\wedge E} \leftarrow i\nabla\times E$$

$$\nabla F = \nabla(\vec{E}+i\vec{B}) =$$

$$= \nabla\cdot E + \nabla\cdot iB + \nabla\wedge E + \nabla\wedge iB$$

$$= \underset{\text{scalar}}{\nabla\cdot E} \underset{\text{Vector}}{- \nabla\times B} + \underset{\text{Bivector}}{i\nabla\times E} + \underset{\text{Trivector}}{i\nabla\cdot B}$$

ⓑ Electromagnetism in ONE Equation:

$$\boxed{\nabla F + \frac{1}{c}\dot{F} = (\rho - \frac{1}{c}\vec{J})}$$

Scalar	$\nabla\cdot E = \rho$
Vector	$-\nabla\times B + \frac{1}{c}\dot{E} = -\frac{1}{c}\vec{J}$
Bivector	$i\nabla\times E + \frac{1}{c}i = 0$
Trivector	$i\nabla\cdot B = 0$

Jancewicz, *Multivectors + Clifford Algebra in Electrodynamics*, World Scientific 1988 pg 78

IV. Physical Applications

A. Unified Language

- 1. Tower of Mathematical Babel
- 2. Example of Derivation of Characteristics
- 3. Other examples

B. Dimensional Democracy

- 1. Clifford Algebra's Automorphism Invariance
- 2. Differential Multiforms
- 3. Papapetrou Equation new derivation

C. Generalized Curvature

- 1. Pan-Dimensional Curvature
- 2. The Metamorphic Connection
- 3. Polygeodesics

Mathematical Tower of Babel!

II.C.1.
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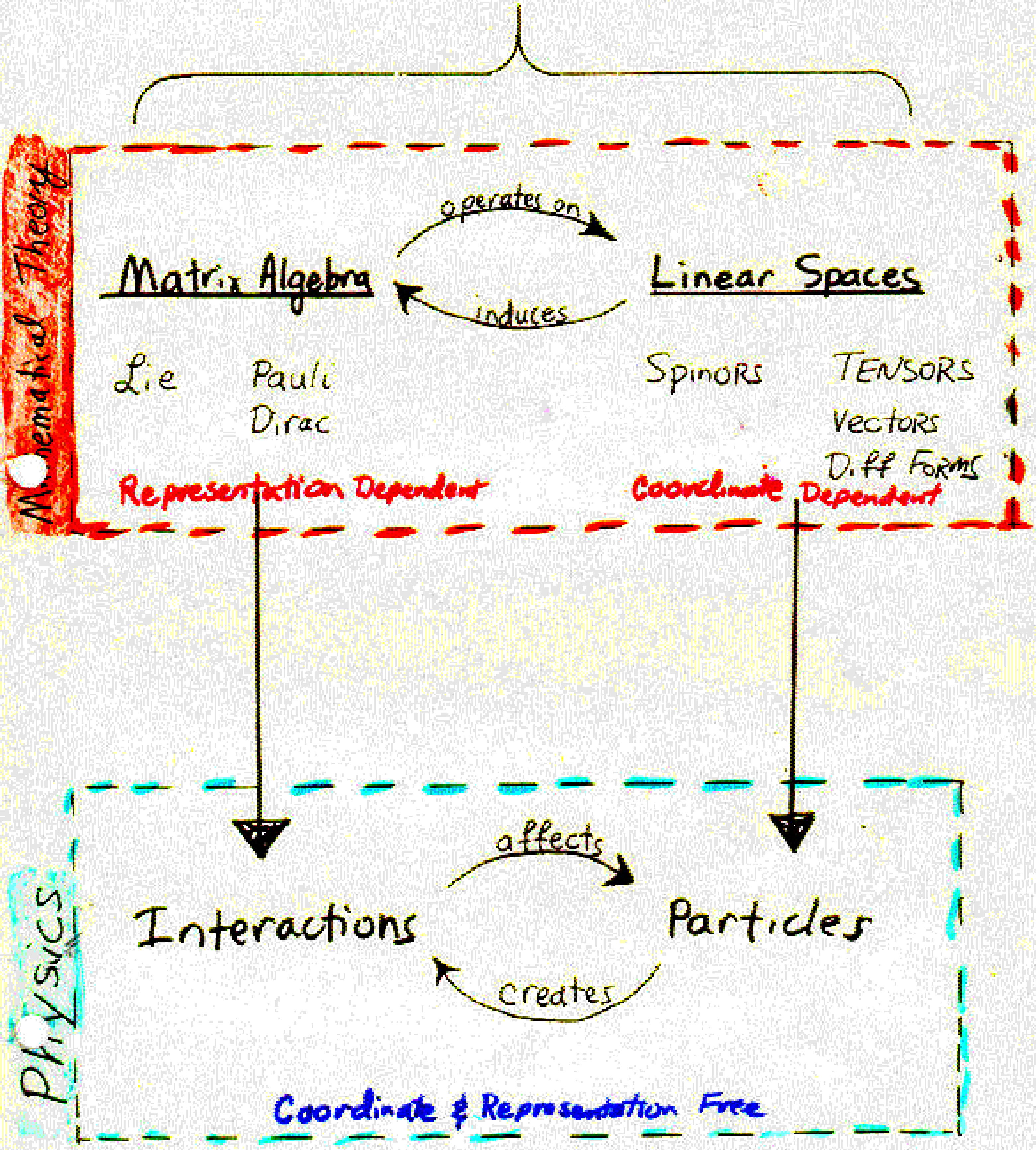
Geometric Algebra

one system does all with less parts!



Mathematical Theory

Matrix Algebra

operates on

induces

Linear Spaces

Lie

Pauli
Dirac

Spinors

TENSORS
Vectors
Diff Forms

Representation Dependent

Coordinate Dependent

Physics

Interactions

affects

creates

Particles

Coordinate & Representation Free

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Comparison of Derivations of Characteristic Hypersurfaces of Maxwell's Equations in 3D

Gibbs Vectors

Alder, Bazin and Schiffer, **Intro to General Relativity** (McGraw-Hill 1965), pp. 108-112.

(4.18) $$\hat{E}(x^1,x^2,x^3) = E(h,x^1,x^2,x^3)$$

(4.19) $$\hat{H}(x^1,x^2,x^3) = H(h,x^1,x^2,x^3)$$

where h is the value of x^0 given by (4.17). The vector functions $\hat{E}$ and $\hat{H}$ are assumed to have continuous first derivatives. Using the above definitions and a bit of vector algebra, we shall be able to obtain a pair of very useful relations (V. Fock, 1959) which the functions E, H, $\hat{E}$, $\hat{H}$, and h must obey on S. These relations will be the key to obtaining the characteristic surfaces of Maxwell's equations.

From (4.18), (4.19), and (4.17) we have

(4.20) $$\frac{\partial \hat{E}_k}{\partial x^i} = \frac{\partial E_k}{\partial x^i} + \frac{\partial E_k}{\partial x^0}\frac{\partial h}{\partial x^i}$$

Setting $k = i$ and summing from 1 to 3, we obtain

(4.21) $$\nabla\cdot\hat{E} = \nabla\cdot E + \frac{1}{c}\dot{E}\cdot\nabla h$$

$$\nabla\cdot\hat{H} = \nabla\cdot H + \frac{1}{c}\dot{H}\cdot\nabla h$$

From (4.20) we also obtain

$$\frac{\partial \hat{E}_k}{\partial x^i} - \frac{\partial \hat{E}_i}{\partial x^k} + \frac{\partial E_k}{\partial x^0}\frac{\partial h}{\partial x^i} - \frac{\partial E_i}{\partial x^0}\frac{\partial h}{\partial x^k} = \frac{\partial E_k}{\partial x^i} - \frac{\partial E_i}{\partial x^k}$$

that is, in vector notation,

(4.24) $$\nabla\times E + \frac{1}{c}\nabla h\times\dot{E} = \nabla\times\hat{E}$$

and as the analogous result for the magnetic field, we have

(4.25) $$\nabla\times H + \frac{1}{h}\nabla h\times\dot{H} = \nabla\times\hat{H}$$

substitute Maxwell's eqns in vac

(4.12) $\nabla\cdot E = 0$

(4.11) $\nabla\times H = \frac{1}{c}\dot{E}$

(4.13) $\nabla\times E = -\frac{1}{c}\dot{H}$

(4.14) $\nabla\cdot H = 0$

Clifford Algebra

Pezzaglia, in Lawrynowicz, Deformations of Mathematical Structures II (1994), pp. 129-134; hep-th/9211062

$$F \equiv E + iH$$

$$\hat{F} \equiv F(h(r), \vec{r})$$

by chain rule

$$\nabla\hat{F} = \nabla F + \frac{1}{c}\nabla h(\partial_t F)$$

$$\nabla F = -\frac{1}{c}\partial_t F$$

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From (4.12) this gives

$$\nabla \cdot \hat{E} = \frac{1}{c}\, \dot{E} \cdot \nabla h \tag{4.22}$$

For the magnetic field we may obtain the analogous result

$$\nabla \cdot \hat{H} = \frac{1}{c}\, \dot{H} \cdot \nabla h \tag{4.23}$$

Substituting into these last two equations from Maxwell's equations in vacuum, (4.11) and (4.13), we obtain

$$-\frac{1}{c}\dot{H} + \frac{1}{c}\nabla h \times \dot{E} = \nabla \times \hat{E} \tag{4.26}$$

$$\frac{1}{c}\dot{E} + \frac{1}{c}\nabla h \times \dot{H} = \nabla \times \hat{H} \tag{4.27}$$

$$\nabla \hat{F} = (\nabla h - 1)\dot{F}$$

The scalar product of (4.26) and (4.27) with ∇h gives

$$-\frac{1}{c}\nabla h \cdot \dot{H} + \frac{1}{c}\nabla h \cdot \nabla h \times \dot{E} = -\frac{1}{c}\nabla h \cdot \dot{H} = \nabla h \cdot \nabla \times \hat{E} \tag{4.28}$$

$$\frac{1}{c}\nabla h \cdot \dot{E} + \frac{1}{c}\nabla h \cdot \nabla h \times \dot{H} = \frac{1}{c}\nabla h \cdot \dot{E} = \nabla h \cdot \nabla \times \hat{H} \tag{4.29}$$

To scalarize the right side, multiply on left by $(\nabla h+1)$

Also the vector product of (4.26) and (4.27) with ∇h gives

$$-\frac{1}{c}(\nabla h \times \dot{H}) + \frac{1}{c}\nabla h \times (\nabla h \times \dot{E}) = \nabla h \times (\nabla \times \hat{E}) \tag{4.30}$$

$$\frac{1}{c}(\nabla h \times \dot{E}) + \frac{1}{c}\nabla h \times (\nabla h \times \dot{H}) = \nabla h \times (\nabla \times \hat{H}) \tag{4.31}$$

Expanding the double cross product and substituting from (4.26) and (4.27), we have

$$\frac{1}{c}\dot{E} - \nabla \times \hat{H} + \frac{1}{c}\nabla h(\nabla h \cdot \dot{E}) - \frac{1}{c}\dot{E}(\nabla h)^2 = \nabla h \times (\nabla \times \hat{E}) \tag{4.32}$$

$$\frac{1}{c}\dot{H} + \nabla \times \hat{E} + \frac{1}{c}\nabla h(\nabla h \cdot \dot{H}) - \frac{1}{c}\dot{H}(\nabla h)^2 = \nabla h \times (\nabla \times \hat{H}) \tag{4.33}$$

Finally, substituting from (4.28) and (4.29), we get

$$\frac{1}{c}\dot{E} - \nabla \times \hat{H} + \nabla h(\nabla h \cdot \nabla \times \hat{H}) - \frac{1}{c}\dot{E}(\nabla h)^2 = \nabla h \times (\nabla \times \hat{E}) \tag{4.34}$$

$$\frac{1}{c}\dot{H} + \nabla \times \hat{E} - \nabla h(\nabla h \cdot \nabla \times \hat{E}) - \frac{1}{c}\dot{H}(\nabla h)^2 = \nabla h \times (\nabla \times \hat{H}) \tag{4.35}$$

Rearrangement now gives the two key relations that we have been working toward:

$$\frac{1}{c}(1 - |\nabla h|^2)\dot{E} = \nabla \times \hat{H} - \nabla h(\nabla h \cdot \nabla \times \hat{H}) + \nabla h \times (\nabla \times \hat{E}) \tag{4.36}$$

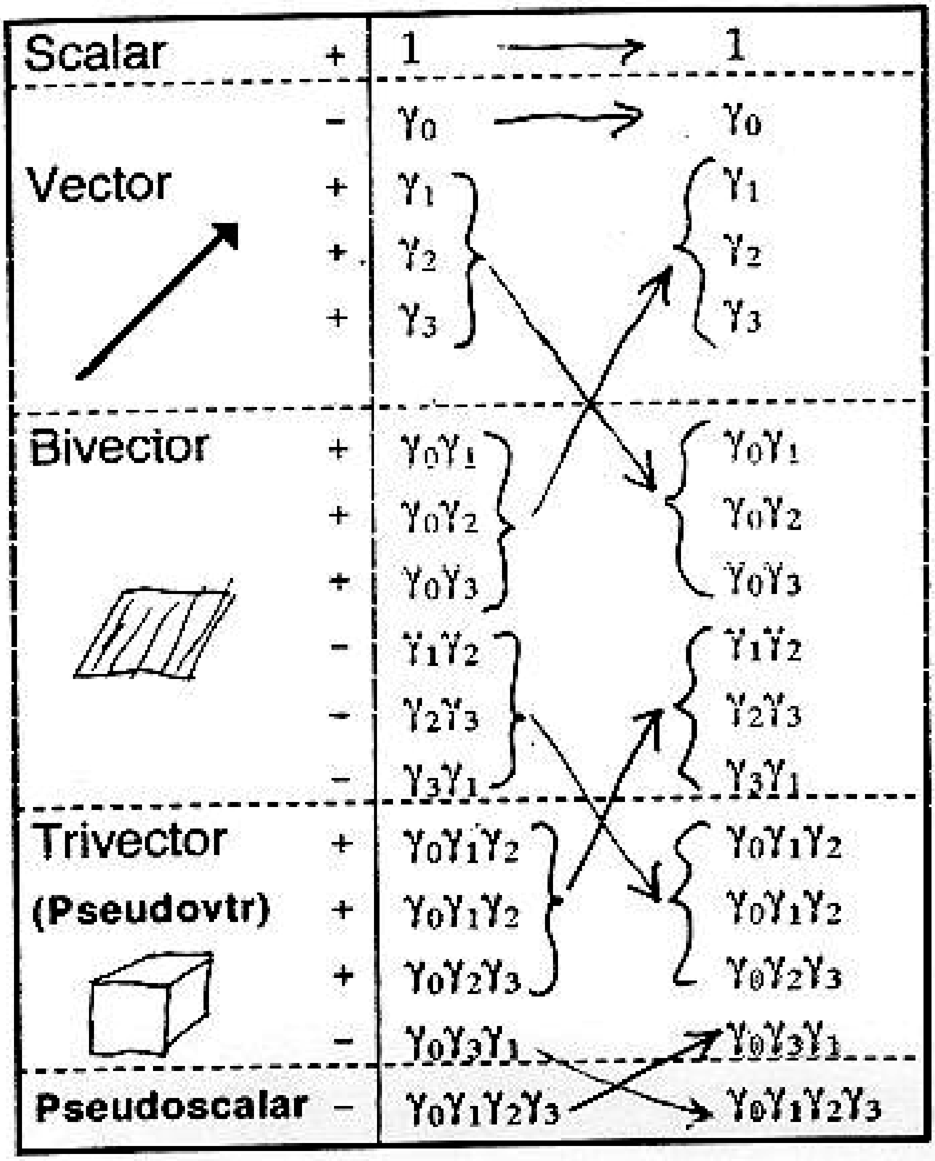
$$\frac{1}{c}(1 - |\nabla h|^2)\dot{H} = -\nabla \times \hat{E} + \nabla h(\nabla h \cdot \nabla \times \hat{E}) + \nabla h \times (\nabla \times \hat{H}) \tag{4.37}$$

$$0 = \nabla \cdot \hat{E} - \nabla h \cdot (\nabla \times \hat{H})$$

$$0 = \nabla \cdot \hat{H} + \nabla h \cdot (\nabla \times \hat{E})$$

$$(\nabla h + 1)\nabla \hat{F} = ((\nabla h)^2 - 1)\dot{F}$$

Automorphism Invariance in 4D



Scalar	+	1	1
	-	γ_0	γ_0
Vector	+	γ_1	γ_1
	+	γ_2	γ_2
	+	γ_3	γ_3
Bivector	+	$\gamma_0\gamma_1$	$\gamma_0\gamma_1$
	+	$\gamma_0\gamma_2$	$\gamma_0\gamma_2$
	+	$\gamma_0\gamma_3$	$\gamma_0\gamma_3$
	-	$\gamma_1\gamma_2$	$\gamma_1\gamma_2$
	-	$\gamma_2\gamma_3$	$\gamma_2\gamma_3$
	-	$\gamma_3\gamma_1$	$\gamma_3\gamma_1$
Trivector	+	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$
(Pseudovtr)	+	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$
	+	$\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$
	-	$\gamma_0\gamma_3\gamma_1$	$\gamma_0\gamma_3\gamma_1$
Pseudoscalar	-	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$

These preserve the algebra:
$\{\gamma_\alpha, \gamma_\beta\}=2g_{\alpha\beta}$
(automorphism)

You can exchange some of the vectors for bivectors and end up with a geometric algebra which is indistinguishable from where you started. You've reshuffled the geometry, sorta a 'rotation' of a vector into a bivector. Should physics be invariant under this type of transformation?

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② Differential Multiforms

Overthrow the Vector Oligarchy - give each geometric element a coordinate!

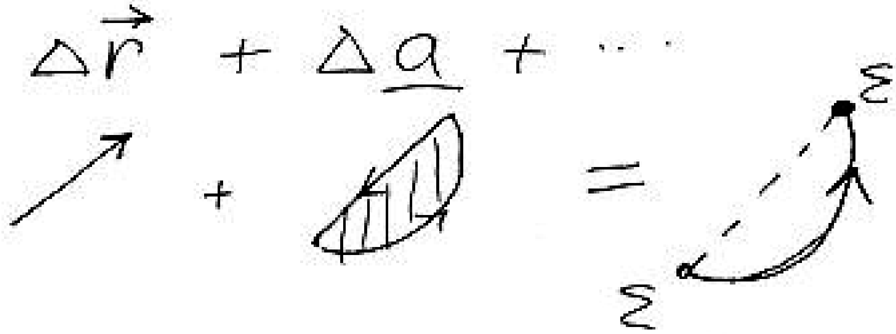
(a) Polydimensional Differential

$$d\Sigma = 1\,d\kappa + d\vec{r} + \tfrac{1}{2} e_u \wedge e_v\, da^{uv} + \cdots$$

scalar + vector + Bivector + ⋯

(b) Multivector Displacement (vector + bivector)

$$\int d\Sigma = \Delta\vec{r} + \Delta\underline{a} + \cdots$$



(c) Differential operator

$$\mathbb{D} f \equiv d\vec{r}\cdot\nabla f + \underbrace{\tfrac{1}{2}\, da^{uv}\frac{\partial f}{\partial a^{uv}}}_{\text{path dependent}} + \cdots$$

Hence $f = f(\vec{r}, \underline{a})$ is a 'path dependent' function.

Argue $\frac{1}{2}\int da^{uv}\frac{\partial f}{\partial a^{uv}} = \oint f$

$$\Rightarrow \boxed{\frac{\partial}{\partial a^{uv}} = \frac{1}{2}\left[\partial_v, \partial_u\right]}$$

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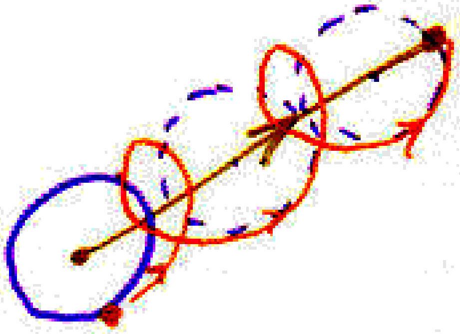
IV. B. 3

III.C Spinning Particles in Curved Space

are known <u>not</u> to follow geodesics in vector space, (Papapetrou eqns below) which violates Einstein's Equivalence Principles "EEP". They <u>do</u> follow "poly geodesics" in the Clifford manifold! (New Result)

1. Velocities



- Vector: "center of mass" $\dot{x}^\nu \equiv \frac{dx^\nu}{d\tau} \simeq \frac{p^\nu}{m}$
- Bivector: "Spin" $\dot{a}^{\alpha\beta} \equiv \frac{da^{\alpha\beta}}{d\tau} \simeq \frac{1}{2}R^2\frac{d\theta^{\alpha\beta}}{d\tau} \equiv \frac{S^{\alpha\beta}}{m}$ Spin Tensor

2. Directional Derivative $e_\alpha = e_\alpha(x^\nu(\tau), a^{\alpha\beta}(\tau))$

- $\dot{e}_\alpha \equiv \frac{de_\alpha}{d\tau} = \dot{x}^\mu \partial_\mu e_\alpha + \frac{S^{\mu\nu}}{2m}[\partial_\mu, \partial_\nu]e_\alpha$
- Affine Connection: $\partial_\mu e_\alpha \equiv \Gamma_{\mu\alpha}{}^\beta e_\beta$
- Curvature: $[\partial_\mu, \partial_\nu]e_\alpha = R_{\mu\nu\alpha}{}^\beta e_\beta$

3. Papapetrou Equations from Polygeodesics

$$0 = \frac{d}{d\tau}(p^\mu e_\mu) = \left(\dot{p}^\mu + p^\nu \dot{x}^\beta \Gamma_{\beta\nu}{}^\mu + \frac{1}{2}\dot{x}^\sigma S^{\alpha\beta} R_{\alpha\beta\sigma}{}^\mu\right)e_\mu$$

$$\underbrace{0 = \frac{d}{d\tau}(S^{\alpha\beta} e_\alpha \wedge e_\beta)}_{\text{Constants of motion}} = \left[\dot{S}^{\alpha\beta} + \dot{x}^\nu\left(S^{\delta\beta}\Gamma_{\nu\delta}{}^\alpha + S^{\alpha\delta}\Gamma_{\nu\delta}{}^\beta\right) + \frac{1}{2m}S^{\mu\nu}\left(S^{\delta\beta}R_{\mu\nu\delta}{}^\alpha + S^{\alpha\delta}R_{\mu\nu\delta}{}^\beta\right)\right]e_\alpha \wedge e_\beta$$

Green is Riemannian Geodesic

Red is spin terms that violate EEP

4

Standard Riemann Space

Affine Connection Preserves Rank

$$\partial_\mu \hat{e}_\nu = \Gamma_{\mu\nu}{}^{\alpha}\hat{e}_\alpha$$

as do geodesics

$$\frac{d}{d\tau}(p^\mu \hat{e}_\mu) = (\dot{p}^\mu + \dot{x}^\alpha p^\nu \Gamma_{\alpha\nu}{}^{\mu})\hat{e}_\mu$$

4½

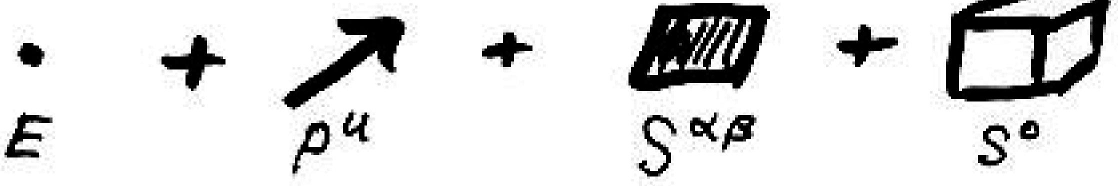
Parallel Transport around a closed loop preserves length, rank, but not direction

$$\Delta V^\nu = R_{\alpha\beta\mu}{}^{\nu} V^\mu \Delta A^{\alpha\beta}$$

$$R_{\mu\nu\alpha}{}^{\beta}\hat{e}_\beta = [\partial_\mu, \partial_\nu]\hat{e}_\alpha$$

Suppose you transported a vector around the loop and it came back as a bivector (plane)?

Requires Objects to be Polygeometric



Requires Laws to be Polygeometric also

$$\frac{d}{d\tau}\left(p^\mu e_\mu + \frac{1}{2}S^{\alpha\beta}e_\alpha \wedge e_\beta\right) = 0$$

② Metamorphic Connection

(a) Local Automorphism invariance - can reshuffle geometry at each point!

(b) Connection will not preserve Rank:

$$\frac{\partial \hat{e}_\mu}{\partial x^\nu} = \underbrace{\Gamma_{\nu\mu}^{\ \ \alpha}\,\hat{e}_\alpha}_{\text{Vector}} + \underbrace{\frac{1}{2}\,\Xi_{\nu\mu}^{\ \ \sigma\omega}\,\hat{e}_\sigma \wedge \hat{e}_\omega}_{\text{Bivector}}$$

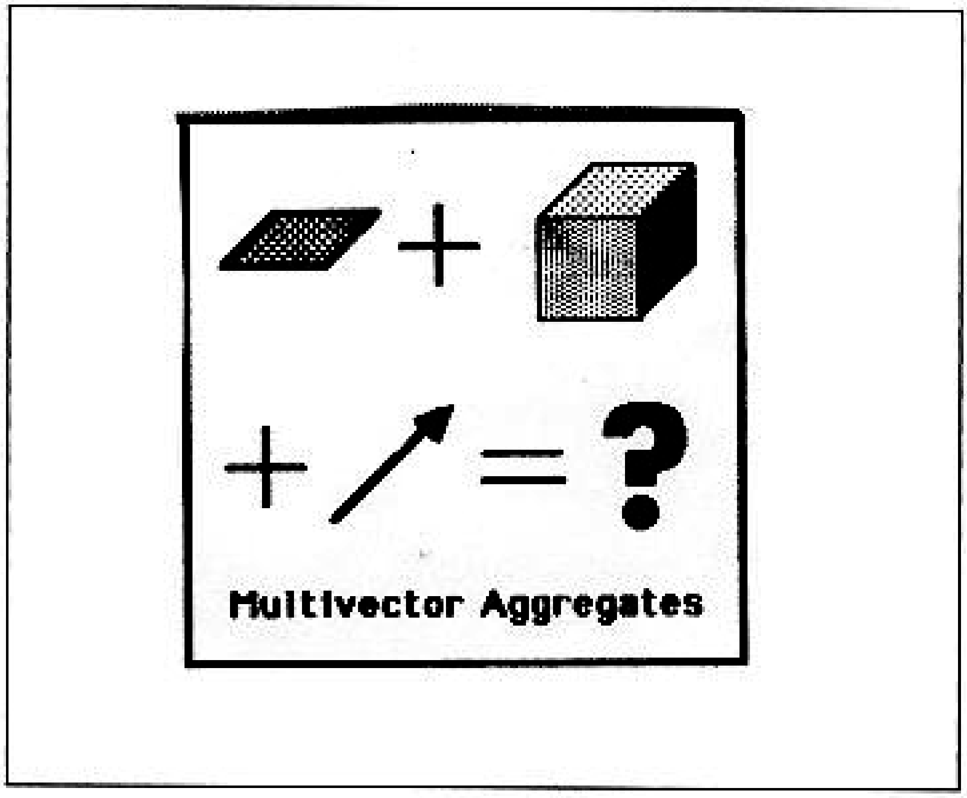
Note Liebnitz rule will not hold for $\wedge$!

$$2\partial_\nu(e_1 \wedge e_2) = [\partial_\nu e_1, e_2] + [e_1, \partial_\nu e_2]$$
$$\neq 2(\partial_\nu e_1) \wedge e_2 + 2\, e_1 \wedge (\partial_\nu e_2)$$

(c) Polygeodesics

- The 'geodesics' (autoparallels) will acquire new terms coupling spin and momentum, i.e. 'new forces'.
- Source of this "new" curvature will be in part due to the spin of particles.
- Calculus of variations is messy - may be non-holonomic

William Marvyn Pezzaglia Jr. (1953-20??)



I propose the following "pan-dimensional" metaprinciples:

- (1) *Relative Dimensionalism*: Dimension is in the eye of the beholder. The geometric rank that an observer assigns to an object (e.g. bivector) is a function of the observer's frame
- (2) *Polydimensional Isotropy:* There is no absolute "direction" to which one can assign the geometry of a vector.
- (3) *Dimensional Democracy:* The laws should be multivectorial in form (having scalar, vector, bivector, parts). Every geometric piece of the equation must be physically realized. Each element of geometry has an associated coordinate.
- (4) *Metamorphic Covariance:* The laws of physics should be form invariant under local automorphism transformations which reshuffle the physical geometry (e.g. trade vectors for bivectors).

"Polydimensional Relativity, a Classical Generalization of hte Automorphism Invariance Principle" (1996) (http://xxx.lanl.gov/abs/gr-qc/9608052)

V. Epagogy

- **Aesthetics**
  - I cannot believe that anything so ugly as the multiplication of matrices is an essential part of the scheme of nature -Sir Arthur Eddington
  - "Beware of Quaternions! They are seductive sirens, always holding out the promise of new and alluring visions of beauty. Remember that many have lost their wits or at least (like I did) several years of their lives in their service. Just when you think you have reached their promised treasure, they slip away". - J.M. Jauch (1973) in a letter warning J. D. Edmonds he was "locked in their clutches".
  - Letter from Hamilton (to Tait) April 12, 1859: "Could anything be simpler or more satisfactory? Don't you feel, as well as think, that we are on a right track, and shall be thanked hereafter? Never mind when".
- **Pragmatic**:
  - "Anyone who has ever used any other parametrization of the rotation group will, within hours of taking up the quaternion parametrization, lament his or her misspent youth" -Simon L. Altmann, *Rotations, Quaternions, and Double Groups* (1986), p.28.
  - May 1893, A. Macfarlane (student of Tait) supports Gibbs & Heaviside's postive square of the vector, calling the others "the minus men". He makes the pragmatic statement: "Thus, the mathematical structure of physics shouldbe dependent on the needs of physics, rather than being imposed from outside".

VI. References

(in order of “complexity”

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