Why does "real" physics need "imaginary" numbers?
A history of physical applications of geometric algebras?

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Abstract

Various mathematical systems: complex algebra, Gibbs vectors (1881) and Sylvester's matrices (1850), have wide applications in physics, engineering and chemistry. However there were many siblings (Hamilton's Quaternions 1843, Grassmann's Algebra of Extension 1844, Cayley's Octonions 1845) that fell into obscure disuse but have recently found new applications which we will review. In particular, William Kingdon Clifford proposed (1876) a universal geometric algebra which combined features of all the above. Due to his untimely death this extraordinary system, in which one can add a vector to a scalar to a plane, was ignored for nearly 100 years, but has a recent revival.

A mathematical language has utility in a discipline (e.g. physics, chemistry) when its structure naturally encodes a principle of that field. For example, Maxwell's equations expressed with quaternions (later with vectors) builds in rotational invariance. Currently physics employs a plethora of languages, each specific to a particular phenomena (e.g. spinors and complex numbers for quantum mechanics, vectors for classical mechanics, tensors for relativity and Lie algebras for field theory). Certainly the expression of a "unified theory of everything" should require a single unified language which embodies all the properties of the above. A possible candidate is Clifford algebra.

Undergraduates can more quickly learn and apply Clifford vectors than Gibbs vectors (or Pauli or Dirac Matrices); becoming quite excited with the interpretation of "i" as the volume of 3 space, seeing "planes" as things that cause rotations (or Lorentz transformations), and being able to do divergence, curl and gradient in a single equation. But there is far more here than mere reformulation or translation. The development of Clifford calculus is still in its infancy. One proposal is to include coordinates for planes and volumes, providing a new derivation of the motion of spinning particles in gravity fields. A generalized curvature in which the connection which does not preserve the rank of a multivector under parallel transport gives a new approach to unified field theory.

This will be a very general talk, of interest to undergraduates (I myself was introduced to the subject as a sophomore, and was captivated by the insights it gave that I found nowhere else)..

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- Updated: 2007Dec26
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Why does “real physics” need “imaginary” numbers?

A history of physical applications of geometric algebras.

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• Shadow of Shadows: History of imaginaries
• Geometric Algebra: Adding a vector to a plane
• Physical Applications: New math => New physics?

1999 Apr 08, Thur 4-5 pm, Washington Square Hall 207 (refreshments McQuarrie Hall 208 at 3:30) San Jose State University, Math Dept Colloquium.
http://www.clifford.org/~wpezzag/talks.html#99apr08
Outline

• I. Introduction

• II. Shadow of Shadows: Division Algebras
  – A. Complex numbers, History, Why "i" in physics?
  – B. Quaternions of Hamilton: Add a scalar to a vector
  – C. Octonions of Cayley

• III. Geometric Algebra
  – A. Gibbs Vectors and the big debate
  – B. Grassmann's Algebra, differential forms
  – C. Clifford Algebra: Add a vector to a plane

• IV. Polydimensional Physics
  – A. Relative Dimensionalism
  – B. Dimensional Democracy
  – C. Metamorphic Covariance

• V. Epagogy

• VI. References
II. Shadow of Shadows
The Division Algebras

A. Complex Algebra

- 1. An Imaginary Tale of “i” as a number
- 2. Algebra and Peacock’s Principle
- 3. Calculus (Analysis) and Conformal Maps

B. Hamilton’s Quaternions (1843)

- 1. Algebra of Scalar + Vector
- 2. Rotations in 3D and 4D
- 3. Quaternionic Analysis

C. Cayley’s Octonions (1881)

- 1. Multiplication Tables
- 2. Non-Associative Algebra
- 3. Applications (rebel without a cause)
II.A. Complex Numbers, Algebra & Analysis

Review the evolution from \( \mathbb{R} \rightarrow \mathbb{C} \)

1. An Imaginary Tale: The History of "i"

An evolution of the concept of "number"

a. **Real Numbers Represent Things**
   
   30,000 BC Cardinal Numbers  Month has 29 or 30 days, Year 12 or 13 months
   3000 BC Rational Numbers  Babylonians divide circle into parts of 60
   572 BC Irrational Numbers  Pythagorean Triangle

b. **Non-Physical Solutions to Real Equations**
   
   250 AD Diophantes rejects negative numbers as "less than nothing"
   1545 Cardan's equation has "sophistic roots" even when cubic roots are real
   1637 Descartes: "imaginary" number as impossibility of geometric construction

c. **Geometric Interpretation of "i"**
   
   1655? John Wallis gives a geometric construction for "i"
   1793 Casper Wessel: Vectors in 2D, "i" rotates by 90 degrees
   1806 Argand rediscovers the 2D concept, Modulus of complex number
III.A.2 Complex Algebra C

How generalize a mathematical language in a physically useful way?

(a) 1833 Peacock's "Principle of Permanence"
Any generalization of an algebraic concept must satisfy the same computational rules.

(b) Utility of C in Classical Physics (Electronics)
Generalize Ohm's law for networks of resistors to include inductors and capacitors.

Ohm's Law \( V = IR \)
Inductors \( V = \dot{I}L \)
Capacitors \( \dot{V} = I/C \)

\( \begin{align*}
V & = \mathbf{I} \mathbf{Z} \\
3 \text{ equations in one!}
\end{align*} \)

Impedance: \( Z = R + i(\omega L - 1/j\omega C) \)

(c) Necessity of C in Quantum Physics?

Heisenberg Quantization (1925)
\[
[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar
\]

Quantum Commutator (position and momentum don't commute—uncertainty principle)

Is \( i \) the geometry associated with Planck's constant?
II.A.3 COMPLEX ANALYSIS

How "design" generalized calculus?

(a) Holomorphic Derivatives (Analytic)
- Want \( dz^n = z^{n-1} \frac{dz}{dz} \)
- Cauchy-Riemann \( F(z) = g(z) + ih(z) \)
  \[
  0 = \frac{dF}{dz} = \frac{dg}{dx} + i \frac{dh}{dx} = \frac{dg}{dy} - i \frac{dh}{dy}
  \]

(b) Fundamental Theorem of Calculus
  \[
  \int_{z_1}^{z_2} \frac{dF}{dz} \, dz = F(z_2) - F(z_1)
  \]
  Path Independent!

(c) Conformal Maps ("politically incorrect")
- Analytic Functions are solutions to Laplace eqn
- Analytic Function of Analytic Function is Analytic
- Map simple solution to complicated one

\[ w = 1/Z \]

0 2 4 6 E volts
B. Quaternions

Sir William Rowan Hamilton
1805-1865

\[ i^2 = j^2 = k^2 = ijk = -1 \]

Famous equation, carved in Brougham Bridge Oct 6, 1843.

1. Anticommutativity is Perpendicularity:

\[ i^2 = j^2 = -1 \]

\[ ij = -ji \]

Define: \( k = ij \) \( \Rightarrow j = ki = -ik \)

As a "Group" \( \mathbb{F} \) \( i, j, k \) describe \( \text{SU}(2) \)

Quaternion: \( H = w1 + xi + yj + zk \)

Adjoint: \( H^+ = w1 - xi - yj - zk \)

Norm: \( H^+ H = w^2 + x^2 + y^2 + z^2 \)
II.A.2 Quaternionic Algebra

(a) Hamilton defines \( H = \omega + \mathbf{\vec{v}} \)

(b) Product of two vectors:
\[
AB = \left( A_1 \hat{e} + A_2 \hat{j} \right) \left( B_1 \hat{e} + B_2 \hat{j} \right) = A_1 B_1 \hat{e}^2 + A_2 B_2 \hat{j}^2 + A_1 B_2 \hat{e} \hat{j} + A_2 B_1 \hat{j} \hat{e} = -(A_1 B_1 + A_2 B_2) + (A_1 B_2 - A_2 B_1) \hat{k}
\]

\[
AB = A \cdot \mathbf{\vec{B}} + A \times \mathbf{\vec{B}}
\]

\[
A \cdot \mathbf{\vec{B}} = -\frac{1}{2} \{A, B\} = -\frac{1}{2} (AB + BA)
\]

\[
A \times \mathbf{\vec{B}} = \frac{1}{2} [A, B] = \frac{1}{2} (AB - BA)
\]

(c) Calculus: Hamilton introduces \( \nabla \) operator:
\[
\nabla = \hat{\imath} \partial_x + \hat{\jmath} \partial_y + \hat{k} \partial_z
\]
\[
\nabla^2 = -\nabla \cdot \mathbf{\vec{E}} + \nabla \times \mathbf{\vec{E}}
\]

(1860) Maxwell's Equations originally written with quaternions:
\[
\nabla \cdot \mathbf{\vec{E}} = -\frac{\partial \mathbf{\vec{B}}}{\partial t}
\]
\[
\nabla \times \mathbf{\vec{E}} = \mathbf{\vec{B}}
\]

These 2 equations encode the 4 Maxwell Equations!
11.3 Rotations

(a) Reflection of a vector \( \vec{V} = (x\hat{i} + y\hat{j} + z\hat{k}) \)
- \( -\hat{i} \vec{V} \hat{i} = x\hat{i} - y\hat{j} - z\hat{k} \)
- \( -\hat{n} \vec{V} \hat{n} \) will reflect \( \vec{V} \) around \( \hat{n} \)

(b) Two Reflections make a rotation:
\( \vec{V}' = b a \vec{V} a b \)

will rotate vector \( \vec{V} \) in plane described by \( \hat{a} \) & \( \hat{b} \)
by angle \( \phi \)

(c) Exponential Form
\(-ba = +a \hat{o} b + a x b \)
\[ = \cos \frac{\phi}{2} + \hat{n} \sin \frac{\phi}{2} \equiv e \]

where \( \hat{n}^2 = -1 \), \( \hat{n} \) is unit vector \( \perp \) to plane described by vectors \( \hat{a} \) & \( \hat{b} \).

\[ \vec{V}' = e^{\frac{\hat{n}\phi}{2}} \vec{V} e^{-\frac{\hat{n}\phi}{2}} \]

Rotation about arbitrary axis \( \hat{n} \)
by angle \( \phi \)
4. Quaternionic Analysis

(a) Functions of quaternion coordinate are really 4D (Cartesian)

\[ H = \omega \hat{1} + x \hat{i} + y \hat{j} + z \hat{k} \]
\[ = \omega + \vec{r} \]
\[ \mathcal{A}(H) = \mathcal{A}(\omega + \vec{r}) = f_0(\omega + \vec{r}) + \vec{f}(\omega + \vec{r}) \]

(b) Differentiation has issues of order due to non-commutativity:

\[ dH^3 = \begin{cases} 3H^2 \frac{dH}{dt} \\ 3dH H^2 \\ 3HDdH \end{cases} \]

(c) Kristenseto p.75 states generalization for Cauchy-Riemann are &

\[ \frac{df_0}{d\omega} = \nabla \cdot \vec{f} \]
\[ -\frac{d\vec{f}}{d\omega} = \nabla f_0 + \nabla \times \vec{f} \]

What about integration? has it been done?
• (d). Quaternions vs. Matrices

"Anyone who has ever used any other parametrization of the rotation group will, within hours of taking up the quaternion parametrization, lament his or her misspent youth" -Simon L. Altmann, *Rotations, Quaternions, and Double Groups* (1986), p.28.

- 1970s(?) Russians realize quaternions faster than matrices for celestial navigation. TOP SECRET (ballistic missiles). Cosmonauts learn quaternions.

- Avoids Gimble Lock (problem with Euler Angles)

- Smaller roundoff error in successive rotations (matrices determinant drifts, must renormalize)

- Interpolation between two states easy (hard with matrices)

(e) 4D Transformations on H = w^1 + xi + yj + zk

Generalization of Conformal Maps for Quaternions (Mobius)

\[
\begin{align*}
\text{Rotations} & : aHc^{-1} \\
\text{Translations} & : H + b \\
\text{Dilation} & : Hs \\
\text{Inversion} & : 1/H
\end{align*}
\]

\[
H' = \left(\frac{aH + b}{cH + d}\right)
\]

- Vahlen Matrix (1902) of Mobius Transformation form a group

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

- Encodes metaprinicples of isotropy and no preferred center to universe. Dilations could represent expansion of universe. What physical principle is associated with inversion?

- Why has there not been a 3D version of Conformal Maps (for electromagnetism, fluid flow) using quaternions?
(4) Quaternionic Analysis

(a) Functions of quaternion coordinate are really 4D (Cartesian)
\[ H = \omega \hat{i} + x \hat{j} + y \hat{j} + z \hat{k} \]
\[ = \omega + \vec{r} \]
\[ \mathcal{G}(H) = \mathcal{G}(\omega + \vec{r}) = \mathcal{F}_0(\omega + \vec{r}) + \mathcal{F}(\omega + \vec{r}) \]

(b) Differentiation has issues of order due to non-commutativity:
\[ dH^2 = \begin{cases} 
3H^2 dH & ? \\
3dH H^2 & ? \\
3HdH H & ? 
\end{cases} \]

(c) Lounesto p.75 states generalization for Cauchy-Riemann are:
\[ \frac{df_0}{d\omega} = \nabla \cdot \vec{f} \]
\[ -\frac{d\vec{f}}{d\omega} = \nabla f_0 + \nabla \times \vec{f} \]

What about integration? has it been done?
Arthur Cayley
1821-1895

As for everything else, so for a mathematical theory: beauty can be perceived but not explained.

- 1845 Octonions
- 1857 Matrix Algebra
- 1858 Quaternions represented by Matrices

1. Cayley Numbers (Octonions)

(a) The Octaves

- 7 Imaginaries
- Commuting Identity
- Elements Anticommutate

\[ e_i e_j + e_j e_i = -2d_{ij} \]

(b) Multiplication Table

- Any two elements generate a quaternion group
- 480 different permutations of multiplication table
- Heptagon Method

\[ e_a e_{a+2^n} = e_{a-2^{n+1}} \]

indices \( 1 \) to \( 7 \)
\( \mod 7 \)
2. Octonionic Algebra

(a) Alternative Algebra

- Non Associative: \(a(bc) = (ab)c\)
- Define Product: \([a,b,c] = (ab)c - a(bc)\)
- Inverses Exist: \([a,a,b] = [a,b,a] = [b,a,a] = 0\)

(b) Hurwitz Theorem (1898)

There are just four normed division algebras, Real (\(\mathbb{R}\)), Complex (\(\mathbb{C}\)), Quaternion (\(\mathbb{H}\)), Octonion (\(\mathbb{O}\))

(b) Octonionic Analysis (Calculus)

Rumor only that someone in Japan has done it.

3. Applications

(a) John von Neumann (1???)

"Nature must make use of them"

(b) What metaprinciple demands non-associativity as natural language?

(c) Many Physics Papers imply useful for:

- Rotations in 7D/8D
- SU(3), Quarks/Gluons ("color")
III. Geometric Algebra

A. Gibbs Vectors (1881)
   • 1. Gibbs Vector Algebra
   • 2. Coordinate free & Isotropic properties
   • 3. The Vector-Quaternion Debate (1891-4)

B. Grassmann’s Algebra
   • 1. Grassmann’s Extended Quantities
   • 2. Hodge Dual, the Dot Product
   • 3. Differential Forms, Stokes Theorem

C. Clifford Algebra
   • 1. The Clifford Group, Matrix Representation
   • 2. Adding Vectors to Planes
   • 3. Geometric Calculus
J. Willard Gibbs (1839-1903)

“America’s First Theoretical Physicist”

appointed Prof. of Math Physics, Yale 1871 without pay until 1880!

“But I do not so much desire to call your attention to the diversity of the applications of multiple algebra, as to the simplicity and unity of its principles. The more we study the subject, the more that we find all that is most useful and beautiful attaching itself to a few central principles. We begin by studying multiple algebras; we end, I think, by studying Multiple Algebra.” - (1885) presidential address given to the American Association for the Advancement of Science.

- Define Dot Product Positive
  \[
  \begin{align*}
  i \cdot i &= +1 \\
  j \cdot j &= +1 \\
  k \cdot k &= +1
  \end{align*}
  \]

- Antisymmetric Cross Product
  \[
  \begin{align*}
  k &= i \times j = -j \times i \\
  i &= j \times k \\
  j &= k \times i
  \end{align*}
  \]

- Algebra is Non-Associative
  \[A \times (B \times C) \neq (A \times B) \times C\]
Why Use Gibbs Vectors in Physics?

Example: Electromagnetism in 3D

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}
\]

\[
\begin{align*}
\partial_t E_z - \partial_z E_y &= -\partial_x B_x \\
\partial_z E_x - \partial_x E_z &= -\partial_y B_x \\
\partial_x E_y - \partial_y E_x &= -\partial_z B_z
\end{align*}
\]

Coord Free

Gibbs vectors encode the metaprinciple of isotropy (of the Universe)

Consider Complex Gibbs Vectors

\[
\vec{F} = \vec{E} + i \vec{B}
\]

\[
\nabla \cdot \vec{F} = \epsilon
\]

\[
\nabla \times \vec{F} - i \partial_t \vec{F} = \vec{J}
\]

\[
\begin{align*}
\nabla \cdot \vec{E} &= \epsilon \\
\nabla \cdot \vec{B} &= 0 \\
\n\nabla \times \vec{E} - \partial_t \vec{B} &= 0 \\
\n\nabla \times \vec{B} - \partial_t \vec{E} &= \vec{J}
\end{align*}
\]

Again we get several equations in one. But what is the principle? Duality?
III.A.3. Vectors versus Quaternions


Maxwell wrote of the quaternion as “a flaming sword”, the virtue of which lay “in enabling us to see the meaning of the question and its solution” which struggling with physical problems. Elsewhere he states that “he had been striving all his life to be freed from the yoke of the Cartesian coordinates, and had found such an instrument in the Hamiltonian quaternions.” In 1890 Maxwell writes (volume 2) “The invention of the calculus of Quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance, with the invention of triple co-ordinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be the greatest use in all parts of science.”

Peter Gurthrie Tait (student of Hamilton) in preface of Hamilton’s Quaternions, says of Gibbs vectors: “... a sort of hermaphrodite monster, compounded of the notations of Hamilton and Grassmann”.

1892, Lord Kelvin in a letter states: “Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way including Clerk Maxwell.”

April 1893, O. Heaviside: “A vector is not a quaternion; it never was, and never will be, and its square is not negative; the supposed proofs are perfectly rotten at the core.” He goes on to give Professor MacAulay, who is a “quaternionist”, some advice, “A difficulty in the way is that he has got used to quaternions. I know what it is, as I was in the quaternionic slough myself once. But I made an effort, and recovered myself, and have little doubt that Prof. MacAulay can do the same.”

May 1893, A. Macfarlane (student of Tait) supports Gibbs & Heaviside’s positive square of the vector, calling the others “the minus men”. “Thus, the mathematical structure of physics should be dependent on the needs of physics, rather than being imposed from outside”.
### Generalized Directed Numbers for Geometric Elements

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Name</th>
<th>Extensive</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Point</td>
<td>Magnitude (scalar)</td>
</tr>
<tr>
<td></td>
<td>Line</td>
<td>Vector (rotor)</td>
</tr>
<tr>
<td></td>
<td>Plane</td>
<td>Bivector (Leaf)</td>
</tr>
<tr>
<td></td>
<td>Volume</td>
<td>Trivector</td>
</tr>
</tbody>
</table>

**Exterior (outer) Product**

- **Antisymmetric:**
  \[ a \wedge b = -b \wedge a \]
  \[ a \wedge a = 0 \]

- **Associative:**
  \[ a \wedge (b \wedge c) = (a \wedge b) \wedge c \]

**Closed (e.g. in 3D)**

\[ a \wedge b \wedge c \wedge d = 0 \]

**Cannot add scalar + vector!**
(2) More on Grassmann

(a) Hodge geometric dual
\[ *A = \text{dual of } A \]
\[ **A = \pm A \]

(b) Dot Product Defined
\[ a \cdot b = *(a \wedge *b) \]

(c) Gibbs Cross Product (3D)
\[ a \times b = *(a \wedge b) \]

(d) Products of geometric objects in 3D
\[ \uparrow \cdot \square = \leftarrow \]
\[ \uparrow \cdot \Box = \square \]
\[ \uparrow \wedge \Box = 0 \quad \text{No 4D object} \]

(e) Cannot do Hamilton's Rotations!
Grassmann Calculus (Cartan 1923)

(a) Electromagnetism in 4D is coordinate free (unlike tensors!), works in curved space!
\[ \begin{align*}
  dF &= 0 \\
  *d*F &= J
\end{align*} \] or \( \begin{cases} 
  \Box \wedge F = 0 \\
  \Box \cdot F = J
\end{cases} \)

(b) Differential Multiforms
\[ \begin{align*}
  \Omega^1 x &= d\Omega^1 = e_1 dx + e_2 dy + e_3 dz \\
  \Omega^2 x &= d\Omega^2 = e_1 e_2 dx dy + e_2 e_3 dy dz + \ldots \\
  \Omega^3 x &= d\Omega^3 = e_1 e_2 e_3 dx dy dz
\end{align*} \]

(c) Generalized Stokes Thm (Fundamental Theorem & Calculus)
\[ \begin{align*}
  \oint D^{n-1} x \cdot \mathbf{F} &= S(\Omega^n x \cdot \nabla) \cdot \mathbf{F} \\
  \oint D^{n-1} x \wedge \mathbf{F} &= S(\Omega^n x \cdot \nabla) \wedge \mathbf{F}
\end{align*} \]

\[ \begin{align*}
  n=3 & \quad \oint da \wedge \mathbf{E} = S(d\Omega^2 \cdot \nabla) \cdot \mathbf{E} = Sd\Omega^2 (\nabla \cdot \mathbf{E}) \\
  n=2 & \quad \oint d\mathbf{r} \cdot \mathbf{E} = S(d\Omega^1 \cdot \nabla) \mathbf{E} = Sda \cdot (\nabla \mathbf{E}) = Sda \cdot (\nabla \mathbf{E})
\end{align*} \]
I hold in fact:

- (1) That small portions of space are in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them.

- (2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

- (3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter, whether ponderable or ethereal.

- (4) That in the physical world nothing else take place but this variation, subject (possibly) to the law of continuity.

"On the Space-Theory of Matter"

1. The Clifford Group

(a) Basis Vectors \( \sigma_i \) anticommute

"Anticommutativity is perpendicularity" - Hamilton

\[
\begin{align*}
\sigma_i \sigma_j &= -\sigma_j \sigma_i \\
\sigma_i \sigma_i &= +1
\end{align*}
\]

\( \left\{ \sigma_i, \sigma_j \right\} = 2 \delta_{ij} \)

(b) The 3D Case is isomorphic to Pauli Group

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Name</th>
<th>Pauli Element</th>
<th>Lie Group</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Scalar</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Vector</td>
<td>( \left{ \sigma_1, \sigma_2 \right} )</td>
<td>( SL(2, \mathbb{C}) )</td>
</tr>
<tr>
<td></td>
<td>Bivector</td>
<td>( \left{ \sigma_i \sigma_j = i \delta_{ij} \right} )</td>
<td>( SU(2) )</td>
</tr>
<tr>
<td></td>
<td>Trivector</td>
<td>( \sigma_1 \sigma_2 \sigma_3 = i )</td>
<td>( U(1) )</td>
</tr>
</tbody>
</table>

(c) Geometric interpretation of "i"

\[ i^2 = (\sigma_1 \sigma_2 \sigma_3)(\sigma_1 \sigma_2 \sigma_3) = -1 \]

Commutes with all elements!

\[ i \sigma_i = \sigma_2 \sigma_3 \]

Mult by \( i \) gives dual!
Two Possible Metric Signatures of 4D Spacetime

<table>
<thead>
<tr>
<th>Signature</th>
<th>&quot;East&quot;</th>
<th>&quot;West Coast&quot;</th>
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- **Count**: - + - 10 6 10
- **Matrix Rep.**: R(4) H(2)

The four generator's signatures determine the rest. Lorentz Group same for both! SL(2,C) (Duals to vectors) does not commute! INEQUIVALENT!
# Matrix Representation vs. Metric

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## Key

- $X(n)$ = $n$ by $n$ matrix, with the components being:
  - Real Numbers
  - Complex Numbers
  - Quaternionic Numbers

- $^{2}X(n)$ = (Two) Block Diagonal Matrix

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2. Properties of Clifford Algebra

(a) Associative Geometric Product \( a(bc) = (ab)c \)

(b) Like Hamilton's Quaternions, separate product into two parts.

\[
ab = a \cdot b + a \wedge b
\]

- \( a \cdot b = \frac{1}{2}(ab + ba) \)
- \( a \wedge b = \frac{1}{2}(ab - ba) \)

(c) Duals constructed by multiplying by the volume element. In 3D \( \mathbb{R}^3 \)

\[
a \wedge b = -i \ a \wedge b
\]

(d) Can do things Grassmann Can't

<table>
<thead>
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<th>Grassmann</th>
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<td>((e_1e_2) \cdot (e_2e_3) = 0)</td>
<td>((e_1e_2)(e_2e_3) = e_1e_3)</td>
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<td>((e_1e_2) \wedge (e_2e_3) = 0)</td>
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Derivative of a Vector $\vec{E}$

$$\nabla E = \nabla \cdot E + \nabla \wedge E$$

scalar + Bivector $\iota \nabla \times E$

$$\nabla F = \nabla (\vec{E} + i\vec{B}) =$$

$$= \nabla \cdot E + \nabla \cdot i\vec{B} + \nabla \wedge E + \nabla \times i\vec{B}$$

$$= \nabla \cdot E - \nabla \times \vec{B} + \iota \nabla \times E + \iota \nabla \cdot \vec{B}$$

scalar Vector Bivector Trivector

Electromagnetism in ONE Equation:

$$\nabla F + \frac{1}{c} \frac{d}{dt} \vec{F} = (\rho - \frac{1}{c^2} \vec{J})$$

Scalar

$$\nabla \cdot E = \rho$$

Vector

$$- \nabla \times \vec{B} + \frac{1}{c} \dot{\vec{E}} = -\frac{1}{c^2} \vec{J}$$

Bivector

$$\iota \nabla \times E + \frac{1}{c} \iota \dot{\vec{B}} = 0$$

Trivector

$$\iota \nabla \cdot \vec{B} = 0$$

Jancewicz, Multivector and Clifford Algebra in Electrodynamics
World Scientific, 1998, pg. 78
(4.18) \[ \frac{\partial E}{\partial t} = \nabla \times B \]
(4.19) \[ \frac{\partial B}{\partial t} = -\nabla \times E \]

where \( \lambda \) is the value of \( z^2 \) given by (4.17). The vector functions \( \mathbf{E} \) and \( \mathbf{H} \) are assumed to have continuous first derivatives. Using the above definitions and a bit of vector algebra, we shall be able to obtain a pair of very useful relations [V. Fock, 1959] which the functions \( \mathbf{E}, \mathbf{H}, \mathbf{E}, \mathbf{H} \), and \( \lambda \) must obey on \( S \). These relations will be the key to obtaining the characteristic surfaces of Maxwell's equations.

From (4.18), (4.19), and (4.17) we have

(4.20) \[ \frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \]

Setting \( \lambda = 1 \) and summing from 1 to 3, we obtain

(4.21) \[ \mathbf{v} \cdot \mathbf{E} = \mathbf{v} \cdot \mathbf{E} + \frac{1}{c} \mathbf{E} \cdot \mathbf{v} \mathbf{h} \]

From (4.12) this gives

(4.22) \[ \mathbf{v} \cdot \mathbf{E} = \frac{1}{c} \mathbf{E} \cdot \mathbf{v} \mathbf{h} \]

For the magnetic field we may obtain the analogous result

(4.23) \[ \mathbf{v} \cdot \mathbf{H} = \frac{1}{c} \mathbf{H} \cdot \mathbf{v} \mathbf{h} \]

From (4.20) we also obtain

(4.24) \[ \mathbf{v} \times \mathbf{E} + \frac{1}{c} \mathbf{v} \mathbf{h} \times \mathbf{E} = \mathbf{v} \times \mathbf{E} \]

and as the analogous result for the magnetic field, we have

(4.25) \[ \mathbf{v} \times \mathbf{H} + \frac{1}{c} \mathbf{v} \mathbf{h} \times \mathbf{H} = \mathbf{v} \times \mathbf{H} \]

Substituting into these last two equations in vacuum, (4.11) and (4.13), we obtain

(4.26) \[ -\frac{1}{c} \mathbf{H} + \frac{1}{c} \mathbf{v} \mathbf{h} \times \mathbf{E} = \mathbf{v} \times \mathbf{E} \]

(4.27) \[ \frac{1}{c} \mathbf{E} + \frac{1}{c} \mathbf{v} \mathbf{h} \times \mathbf{H} = \mathbf{v} \times \mathbf{H} \]

The scalar product of (4.26) and (4.27) with \( \mathbf{v} \) \( \mathbf{h} \) gives

(4.28) \[ -\frac{1}{c} \mathbf{v} \mathbf{h} \cdot \mathbf{H} + \frac{1}{c} \mathbf{v} \mathbf{h} \cdot \mathbf{v} \mathbf{h} \times \mathbf{E} = -\frac{1}{c} \mathbf{v} \mathbf{h} \cdot \mathbf{H} = \mathbf{v} \mathbf{h} \cdot \mathbf{v} \times \mathbf{H} \]

(4.29) \[ \frac{1}{c} \mathbf{v} \mathbf{h} \cdot \mathbf{E} + \frac{1}{c} \mathbf{v} \mathbf{h} \cdot \mathbf{v} \mathbf{h} \times \mathbf{H} = \frac{1}{c} \mathbf{v} \mathbf{h} \cdot \mathbf{E} = \mathbf{v} \mathbf{h} \cdot \mathbf{v} \times \mathbf{H} \]

Also the vector product of (4.26) and (4.27) with \( \mathbf{v} \) \( \mathbf{h} \) gives

(4.30) \[ -\frac{1}{c} (\mathbf{v} \mathbf{h} \times \mathbf{H}) + \frac{1}{c} \mathbf{v} \mathbf{h} \times (\mathbf{v} \mathbf{h} \times \mathbf{E}) = \mathbf{v} \mathbf{h} \times (\mathbf{v} \times \mathbf{E}) \]

(4.31) \[ \frac{1}{c} (\mathbf{v} \mathbf{h} \times \mathbf{E}) + \frac{1}{c} \mathbf{v} \mathbf{h} \times (\mathbf{v} \mathbf{h} \times \mathbf{H}) = \mathbf{v} \mathbf{h} \times (\mathbf{v} \times \mathbf{H}) \]

Expanding the double cross product and substituting from (4.26) and (4.27), we have

(4.32) \[ \frac{1}{c} \mathbf{E} - \mathbf{v} \times \mathbf{H} + \frac{1}{c} \mathbf{v} \mathbf{h} \times (\mathbf{v} \mathbf{h} \cdot \mathbf{E}) - \frac{1}{c} \mathbf{E} (\mathbf{v} \mathbf{h}) = \mathbf{v} \mathbf{h} \times (\mathbf{v} \times \mathbf{E}) \]

(4.33) \[ \frac{1}{c} \mathbf{H} + \mathbf{v} \times \mathbf{E} + \frac{1}{c} \mathbf{v} \mathbf{h} \times (\mathbf{v} \mathbf{h} \cdot \mathbf{H}) - \frac{1}{c} \mathbf{H} (\mathbf{v} \mathbf{h}) = \mathbf{v} \mathbf{h} \times (\mathbf{v} \times \mathbf{H}) \]

Finally, substituting from (4.23) and (4.29), we get

(4.34) \[ \frac{1}{c} \mathbf{E} - \mathbf{v} \times \mathbf{H} + \mathbf{v} \mathbf{h} \times (\mathbf{v} \mathbf{h} \cdot \mathbf{H}) - \frac{1}{c} \mathbf{E} (\mathbf{v} \mathbf{h}) = \mathbf{v} \mathbf{h} \times (\mathbf{v} \times \mathbf{E}) \]

(4.35) \[ \frac{1}{c} \mathbf{H} + \mathbf{v} \times \mathbf{E} - \mathbf{v} \mathbf{h} \times (\mathbf{v} \mathbf{h} \cdot \mathbf{E}) - \frac{1}{c} \mathbf{H} (\mathbf{v} \mathbf{h}) = \mathbf{v} \mathbf{h} \times (\mathbf{v} \times \mathbf{H}) \]

Rearrangement now gives the two key relations that we have been working toward:

(4.36) \[ \frac{1}{c} (1 - [\mathbf{v} \mathbf{h}]^2) \mathbf{E} = \mathbf{v} \times \mathbf{H} - \mathbf{v} \mathbf{h} (\mathbf{v} \mathbf{h} \cdot \mathbf{H}) + \mathbf{v} \mathbf{h} \times (\mathbf{v} \times \mathbf{E}) \]

(4.37) \[ \frac{1}{c} (1 - [\mathbf{v} \mathbf{h}]^2) \mathbf{H} = \mathbf{v} \times \mathbf{E} + \mathbf{v} \mathbf{h} (\mathbf{v} \mathbf{h} \cdot \mathbf{E}) + \mathbf{v} \mathbf{h} \times (\mathbf{v} \times \mathbf{H}) \]
Comparison of Derivations of Characteristic Hypersurfaces of Maxwell’s Equations in 3D

Gibbs Vectors


\[
\begin{align*}
\mathbf{E}(x^1, x^2, x^3) &= E(x^1, x^2, x^3) \\
\mathbf{A}(x^1, x^2, x^3) &= A(x^1, x^2, x^3)
\end{align*}
\]

where \( \mathbf{E} \) is the value of \( \mathbf{E} \) given by (4.17). The vector functions \( \mathbf{E} \) and \( \mathbf{A} \) are assumed to have continuous first derivatives. Using the above definitions and a bit of vector algebra, we shall be able to obtain a pair of very useful relations (V. Fock, 1959) which the functions \( \mathbf{E}, \mathbf{H}, \mathbf{\dot{E}}, \mathbf{\dot{H}} \) and \( \mathbf{v} \) must obey on \( S \). These relations will be the key to obtaining the characteristic surfaces of Maxwell’s equations.

From (4.18), (4.19), and (4.17) we have

\[
\begin{align*}
\frac{\partial E_i}{\partial x^j} = \frac{\partial E_j}{\partial x^i} + \frac{\partial H_i}{\partial x^j} - \frac{\partial H_j}{\partial x^i}
\end{align*}
\]

Setting \( k = i \) and summing from 1 to 3, we obtain

\[
\begin{align*}
\begin{align*}
\nabla \cdot \mathbf{E} &= \nabla \cdot \mathbf{E} + \frac{1}{c} \nabla \cdot \mathbf{A} \\
\nabla \cdot \mathbf{H} &= \nabla \cdot \mathbf{H} + \frac{1}{c} \nabla \cdot \mathbf{A}
\end{align*}
\end{align*}
\]

From (4.20) we also obtain

\[
\begin{align*}
\frac{\partial E_i}{\partial x^j} - \frac{\partial E_j}{\partial x^i} + \frac{\partial E_i}{\partial x^j} - \frac{\partial E_j}{\partial x^i} = \frac{\partial V}{\partial x^i} - \frac{\partial V}{\partial x^j}
\end{align*}
\]

that is, in vector notation,

\[
\nabla \times \mathbf{E} + \frac{1}{c} \nabla \times \mathbf{A} = \nabla \times \mathbf{E}
\]

and as the analogous result for the magnetic field, we have

\[
\nabla \times \mathbf{H} + \frac{1}{c} \nabla \times \mathbf{A} = \nabla \times \mathbf{H}
\]

Substitute Maxwell’s Eqns in vac

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \times \mathbf{H} &= \frac{1}{c} \mathbf{E} \\
\nabla \cdot \mathbf{E} &= -\frac{1}{c} \mathbf{A} \\
\n\nabla \cdot \mathbf{A} &= 0
\end{align*}
\]

Clifford Algebra

Pezzaglia, in Lawrynowicz, Deformations of Mathematical Structures II (1994), pp. 129-134; hep-th/9211062

\[
\begin{align*}
\mathbf{F} &= \mathbf{E} + i \mathbf{A} \\
\mathbf{\nabla} \mathbf{F} &= \mathbf{\nabla} \mathbf{E} + \frac{i}{c} \mathbf{\nabla} \mathbf{A}
\end{align*}
\]

by chain rule

\[
\nabla \mathbf{F} = \nabla \mathbf{E} + \frac{1}{c} \nabla \mathbf{A} (\nabla \mathbf{E})
\]

\[
\nabla \mathbf{F} = -\frac{1}{c} \partial_c \mathbf{F}
\]
From (4.12) this gives

$$\nabla \cdot E = \frac{1}{c} \frac{1}{c} E \cdot \nabla A$$

For the magnetic field we may obtain the analogous result

$$\nabla \cdot H = \frac{1}{c} \frac{1}{c} H \cdot \nabla h$$

Substituting into these last two equations from Maxwell's equations in vacuum, (4.11) and (4.13), we obtain

$$-\frac{1}{c} \nabla \cdot H + \frac{1}{c} \nabla \times E = \nabla \times \vec{E}$$

$$\frac{1}{c} \vec{E} + \frac{1}{c} \nabla \times H = \nabla \times \vec{H}$$

The scalar product of (4.26) and (4.27) with \nabla h gives

$$-\frac{1}{c} \nabla \cdot H + \frac{1}{c} \nabla \times E = -\frac{1}{c} \nabla \cdot H = \nabla \cdot \nabla \times \vec{E}$$

$$\frac{1}{c} \nabla \cdot \vec{E} + \frac{1}{c} \nabla \times \nabla \times H = \frac{1}{c} \nabla \cdot \vec{E} = \nabla \cdot \nabla \times \vec{H}$$

Also the vector product of (4.26) and (4.27) with \nabla h gives

$$-\frac{1}{c} \nabla \times H + \frac{1}{c} \nabla \times (\nabla \times \vec{E}) = \nabla \times (\nabla \times \vec{E})$$

$$\frac{1}{c} \nabla \times \vec{E} + \frac{1}{c} \nabla \times (\nabla \times H) = \nabla \times (\nabla \times \vec{H})$$

Expanding the double cross product and substituting from (4.26) and (4.27), we have

$$\frac{1}{c} \vec{E} - \nabla \times \vec{H} + \frac{1}{c} \nabla \cdot (\nabla \times \vec{E}) - \frac{1}{c} \vec{E} (\nabla \times \vec{E}) = \nabla \times (\nabla \times \vec{E})$$

$$\frac{1}{c} \vec{H} + \nabla \times \vec{E} + \frac{1}{c} \nabla \cdot (\nabla \times \vec{H}) - \frac{1}{c} \vec{H} (\nabla \times \vec{H}) = \nabla \times (\nabla \times \vec{H})$$

Finally, substituting from (4.28) and (4.29), we get

$$\frac{1}{c} \vec{E} - \nabla \times \vec{H} + \nabla \cdot (\nabla \times \vec{E}) - \frac{1}{c} \vec{E} (\nabla \times \vec{E}) = \nabla \times (\nabla \times \vec{E})$$

$$\frac{1}{c} \vec{H} + \nabla \times \vec{E} + \frac{1}{c} \nabla \times (\nabla \times H) = \nabla \times (\nabla \times \vec{H})$$

Rearrangement now gives the two key relations that we have been working toward:

$$\frac{1}{c} (1 - (\nabla h)^2) \vec{E} = \nabla \times \vec{H} - \nabla \cdot (\nabla \times \vec{E}) + \nabla \times (\nabla \times \vec{E})$$

$$\frac{1}{c} (1 - (\nabla h)^2) \vec{H} = -\nabla \times \vec{E} + \nabla \times (\nabla \times \vec{H}) + \nabla \times (\nabla \times \vec{E})$$

$$\vec{E} = \nabla \cdot \vec{H} = \nabla \times (\nabla \times \vec{E})$$

$$\vec{H} = \nabla \times \vec{E} = \nabla \times (\nabla \times \vec{H})$$

$$\nabla \hat{F} = (\nabla h - 1) \hat{F}$$

To scalarize the right side, multiply on left by \((\nabla h + 1)\)
IV. Polydimensional Physics
What new metaprinciples might Clifford Algebra embody?

A. Relative Dimensionalism
   1. Review Special Relativity: Scalar+Vector
   2. Automorphism Invariance
   3. Polydimensional Isotropy

B. Dimensional Democracy
   1. Coordinates for All Multivectors
   2. Differential Multiforms
   3. New Classical Action Principle

C. Generalized Curvature
   1. Bivector Derivatives and Curvature
   2. Papapetrou and nonholonomic Calculus
   3. Metamorphic Curvature
A. Relative Dimensionalism

[Apologies to Dr. Who and the Tardis]

1. Special Relativity Review

(a) Unify phenomena with 4th dimension
- Combine Scalar law with vector law with 4-vectors
- Scalar: \( \dot{\mathbf{E}} = e \mathbf{E} \cdot \mathbf{v} \)
- Vector: \( \dot{\mathbf{p}} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \)
- Whether energy is a scalar or fourth component of a vector is relative to your point of view (3D or 4D reference frame).

(b) Metapriniciple: Velocity is Relative
- Hence laws must be invariant under Lorentz Transformations
- What is "time" (scalar) to one observer is combination of space (vector) and time (scalar) to another.

(c) Invariants (Length of 4 vector)
- \( \rho_{\mu} \rho^{\mu} = \frac{E^2}{c^2} - \mathbf{P}^2 = \left(\frac{m_0 c}{m}\right)^2 \)
- \( (\text{scalar})^2 - (\text{vector})^2 = \text{Rest mass} \)
- Motion increases mass \( m = m_0 \sqrt{1 + \left(\frac{\mathbf{P}}{m_0 c}\right)^2} \)
### 2. Automorphism Invariance

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<td></td>
<td>+</td>
<td>$\gamma_0 \gamma_2 \gamma_3$</td>
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<td>-</td>
<td>$\gamma_0 \gamma_1 \gamma_2 \gamma_3$</td>
<td>$\gamma_0 \gamma_1 \gamma_2 \gamma_3$</td>
</tr>
<tr>
<td>Pseudoscalar</td>
<td>-</td>
<td>$\gamma_0 \gamma_1 \gamma_2 \gamma_3$</td>
<td>$\gamma_0 \gamma_1 \gamma_2 \gamma_3$</td>
</tr>
</tbody>
</table>

Physics should be invariant under a change in representation of the Dirac Matrices or equivalently, under a change in spinor basis. These preserve the algebra:

$\gamma_\alpha \cdot \gamma_\beta = 2g_{\alpha\beta}$ (automorphism)

- Above: Rotation between Vector & Bivector (should physics be invariant under this?)
- Duality would be exchange of vector with trivector (pseudovector). That transformation related to Parity and Time Reversal symmetries.
3. Polydimensional Isotropy

(a) **Unify phenomena with Clifford Algebra**

\[ \dot{p}^u = \frac{e}{m} p_v F^{uv} \]

\[ \dot{S}^{u\beta} = \frac{e}{m} (F^v_{\nu} S^{v\beta} - F^\beta_{\nu} S^{v\nu}) \]

\[ \dot{M} = \frac{e}{2m} [M, F] \]

- Polymomenta: \( M = p^u e_\mu + \frac{1}{2} S^{uv} e_\mu \land e_\nu \)

- Electromagnetic Field Bivector: \( F = \frac{1}{2} F^{uv} e_\mu \land e_\nu \)

(b) **Automorphism Invariance**

- Rotation between Bivector and Vector leaves unchanged

- Invariant:

\[ \rho^2 - S^2 = (M_0 c)^2 \]

\[ (\text{Vector})^2 - (\text{Bivector})^2 = (\text{Scalar})^2 \]

- Mass enhanced by spin

\[ M = M_0 \sqrt{1 + \left(\frac{S}{M_0 c \lambda}\right)^2} \]

(c) **Dimension is in the eye of the beholder**

- No "preferred direction" can be associated with vector as opposed to bivector.

- What is vector to one observer is part vector, part bivector in another frame of reference

- The Laws of Physics must be form invariant under (restricted) global automorphism transformations
1. Coordinate for each geometric element
\[ d\Sigma = dx^u \wedge e_u + \frac{1}{2a} da^{uv} \wedge e_u \wedge e_v + \cdots \]

Vector: \( a \)
Bivector: \( b \)

2. Propose New 'path length'
New affine parameter
\[ dk^2 = dx^u dx_u - \frac{1}{2a^2} da^{uv} da_{uv} \]
length\(^2\) - area\(^2\)

3. New Classical Action Principle
Particles follow paths (polygeodesics) which extremize length swept out by momentum plus area swept out by spin.

Lagrangian
\[ L = m_o c \sqrt{x^u x_u - \frac{1}{2a^2} \hat{a}^{uv} \hat{a}_{uv}} \]

Yields new definitions of momenta with spin enhanced mass.

\[ p_u = \frac{\delta L}{\delta \dot{x}^u} = m_o \frac{dx_u}{dt} = m \frac{dx_u}{d\tau} \]

\[ S_{uv} = \lambda^2 \frac{\delta L}{\delta \dot{a}^{uv}} = m_o \dot{a}_{uv} = \lambda m a_{uv} \]

\[ m = m_o \frac{d\tau}{d\kappa} = m_o \sqrt{1 + \left( \frac{S}{m_o \lambda c} \right)^2} \]
C. Curved Space

1. Bivector Derivative
   - Connection \( \frac{\partial e_m}{\partial x^\alpha} = \Gamma^B_{\alpha \mu} e_B \)

* Propose \( \frac{\partial}{\partial a^\alpha_{AB}} = [\partial x^\alpha, \partial x^\beta] - \Gamma^{\sigma}_{\alpha \rho} \partial x^\rho \)

- Hence \( \frac{\partial e_m}{\partial a^\alpha_{AB}} = (R^\sigma_{\alpha B u} - \Gamma^\sigma_{\alpha B} \Gamma^u_{\nu} \sigma) e_{\nu} \)

2. Nonholonomic Calculus of Variations

\[ \delta L = \left( \frac{\delta L}{\delta x^\alpha} \delta x^\alpha + \frac{\delta L}{\delta \dot{x}^\alpha} \delta \dot{x}^\alpha \right) + \frac{1}{2} \left( \frac{\delta^2 L}{\delta a^\alpha_{AB} \delta a^\alpha_{CD}} + \frac{\delta^2 L}{\delta a^\alpha_{CD} \delta a^\alpha_{AB}} \right) \]

\[ \rho^u \delta \dot{x}^{\mu} = \frac{d}{dx} \left( \rho^u \delta x^\mu \right) - \rho^u \delta x^\mu + \rho^u \left( \frac{\delta^2 x^\mu}{\delta s^\alpha} \frac{d}{ds} \delta x^\alpha \right) \]

Boundary Does not Vanish!

\( \left( \delta x^\sigma - \frac{d}{ds} \delta x^\sigma \right) = \delta x^\alpha \dot{x}^\beta \Gamma^\sigma_{\alpha \beta} + \frac{1}{2} (\delta x^\alpha \dot{x}^\mu - \dot{x}^\alpha \delta x^\mu) R^\sigma_{\mu \lambda \nu} \) - torsion

Coupled to Spin

With this I have been able to provide a derivation of the Papapetrou equations from the polydimensional Lagrangian (open problem for 50 years—spinning particles do not follow geodesics, hence violate Einstein's Equivalence principles).
3. Generalized Ideas

(a) \[ \mathcal{L} = \sqrt{\text{distance}^2 + \text{area}^2 + \text{volume}^2 + \ldots} \]
- Points sweep out distance
- Strings sweep out area
- Membranes sweep out volume.

(b) Quantize Classical Spin
\[ [\alpha_{\alpha\beta}, \hat{S}_{\alpha\beta}] = \mathbb{i} \hbar \]
Generalized Dirac Equation
\[ (\gamma^u \hat{p}_u + \frac{1}{2} \gamma^{uv} \hat{S}_{uv} - m_0) \psi = 0 \]

(c) Metamorphic Curvature
- Suppose you parallel transport a vector around a loop and it comes back a different plane?
- Metamorphic Connection!
\[ \frac{de_u}{dx^v} = \Gamma^\alpha_{\nu\mu} e_\alpha + Q^\alpha_{\nu\mu} e_\alpha e_\beta e_\gamma \]
- Paths are polygoadesics, with new forces which couple spin & momenta. New approach to unified field theory.
V. Epagogy

• Aesthetics
  
  – I cannot believe that anything so ugly as the multiplication of matrices is an essential part of the scheme of nature -Sir Arthur Eddington

  – “Beware of Quaternions! They are seductive sirens, always holding out the promise of new and alluring visions of beauty. Remember that many have lost their wits or at least (like I did) several years of their lives in their service. Just when you think you have reached their promised treasure, they slip away”. - J.M. Jauch (1973) in a letter warning J. D. Edmonds he was “locked in their clutches”.

  – Letter from Hamilton (to Tait) April 12, 1859: “Could anything be simpler or more satisfactory? Don’t you feel, as well as think, that we are on a right track, and shall be thanked hereafter? Never mind when”.

• Pragmatic
  
  – “Anyone who has ever used any other parametrization of the rotation group will, within hours of taking up the quaternion parametrization, lament his or her misspent youth” -Simon L. Altmann, Rotations, Quaternions, and Double Groups (1986), p.28.

  – May 1893, A. Macfarlane (student of Tait) supports Gibbs & Heaviside’s postive square of the vector, calling the others “the minus men”. He makes the pragmatic statement: “Thus, the mathematical structure of physics shouldbe dependent on the needs of physics, rather than being imposed from outside”.

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VI. References

(in order of "complexity"


- David Hestenes, Space-Time Algebra, (Gordon and Breach 1966). This is a short book, a good introduction for advanced students.


- Hermann Grassmann, A New Branch of Mathematics (Open Court 1995), This is a translation of his works, which is interesting, but very very complicated.