Equations of Motion of Spinning Particles (in Gravity)

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- **Spinning Particles are non-geodesic, violate EEP**
- **New Action Principle for Classical Spin where $\delta\dot{\mathbf{x}} \neq d\,\delta\mathbf{x}/dt$**
- **Spin contributions to source of gravity**

Pacific Coast Gravity Meeting, ITP, UCSB, 2001Mar10, 11:30-11:45 am

http://www.clifford.org/~wpezzag/talks.html

I. Outline

II.A Hierarchal Handwaving

$3 \oplus 1$ Special Relativity

$$(mc)^2 = \mathcal{E}^2/c^2 - \vec{p}^{\,2}$$

invariant rest mass

(scalar)^2 − (vector)^2

Motion increases Mass $$m' = m\sqrt{1 + \left(\frac{\vec{p}}{mc}\right)^2}$$

Approx: $$m' \simeq m + \frac{1}{2}\frac{\vec{p}^{\,2}}{mc^2}$$

4-vector $$p = p^u e_u$$

invariant $$(mc)^2 = \|p\| = p_u p^u$$

Classical Spin Mechanics

Dixon (70) Mass is Dynamic

$$(m_0 c)^2 = p_u p^u - \frac{S^{uv}S_{uv}}{2\lambda^2}$$

invariant mass

(vector)^2 − (Bivector)^2

Spin increases mass $$m = m_0\sqrt{1 + \left(\frac{|S|}{m_0 c\lambda}\right)^2}$$

Approx: $$m \simeq m_0 + \frac{1}{2}\frac{|S|^2}{m_0 \lambda^2 c^2}$$

$\lambda \simeq$ Radius of gyration

Polyvector: $$\mathfrak{M} = p^u e_u + \frac{1}{2\lambda} S^{uv} e_u \wedge e_v$$

invariant:

$$|\tilde{\mathfrak{M}}\mathfrak{M}| = p_u p^u - \frac{S^{uv}S_{uv}}{2\lambda^2}$$

II.B. USE Geometry to Unify Equations

1. Unify 3D Electrodynamics with 4D

$$\left.\begin{aligned}&\text{- Scalar: } \dot{\mathcal{E}} = e\vec{E}\cdot\vec{v}\\ &\text{- Vector: } \dot{\vec{p}} = e(\vec{E}+\vec{v}\times\vec{B})\end{aligned}\right\} \quad \dot{p}^\mu = \frac{e}{m}p_\nu F^{\mu\nu}$$

2. Coordinate Independent Form

$$p = p^\mu \hat{e}_\mu$$

$$F = \frac{1}{2}F^{\mu\nu}\hat{e}_\mu\wedge\hat{e}_\nu$$

$$\boxed{\dot{p} = \frac{e}{2m}[p, F]}$$

where $[e_\mu, e_\nu] = 2e_\mu\wedge e_\nu$, $[e_\alpha, e_\mu\wedge e_\nu] = 2e_\alpha\cdot(e_\mu\wedge e_\nu)$

3. Unify Spin and Momentum Equations

$$\left.\begin{aligned}&\dot{p}^\mu = \frac{e}{m}p_\nu F^{\mu\nu}\\ &\dot{S}^{\mu\nu} = \frac{e}{m}(F^\mu{}_\lambda S^{\lambda\nu} - F^\nu{}_\lambda S^{\lambda\mu})\end{aligned}\right\}$$

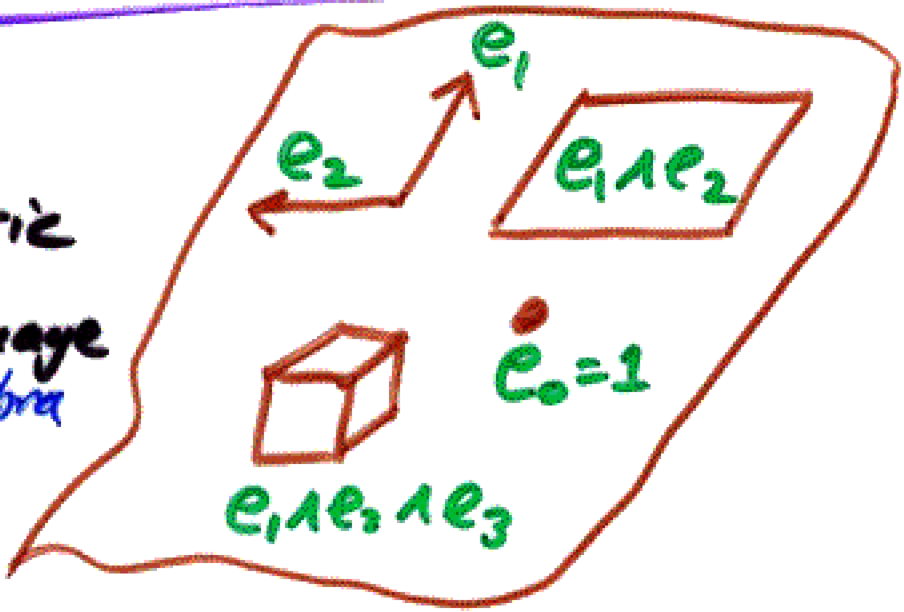
$$\boxed{\dot{\mathcal{M}} = \frac{e}{2m}[\mathcal{M}, F]}$$

Polymomenta: $\mathcal{M} = p^\mu e_\mu + \frac{1}{2\lambda^2}S^{\mu\nu}e_\mu\wedge e_\nu$

II.C Polydimensionalism

① Space is Polygeometric

Natural Language
Clifford Algebra



② Dimensional Democracy

Each geometric element has a Coordinate

$$d\Sigma = d\phi\, \hat{e}_0 + dx^{\mu}\hat{e}_{\mu} + \tfrac{1}{2}\, da^{\mu\nu}\hat{e}_{\mu}\wedge\hat{e}_{\nu} + \cdots$$

Event   scalar   vector   Bivector

③ Conserved Quantities Associated:

Spin : $S^{\mu\nu} = m\,\dot{a}^{\mu\nu}$

Momentum : $p^{\mu} = m\,\dot{x}^{\mu}$

(Mass : $m \simeq \dot{\phi}$ Pavsic hep-th/0011216)

III.A. New Affine Parameter for Mechanics

① Define $$d\kappa^2 = dx^\mu dx_\mu - \frac{1}{2\lambda^2} da^{\mu\nu} da_{\mu\nu}$$

$(\text{modulus})^2 = (\text{length})^2 - (\text{area})^2$

Total Derivative $$\mathring{Q} \equiv \frac{dQ}{d\kappa} = \left(\frac{d\tau}{d\kappa}\right)\frac{dQ}{d\tau} = \left(\frac{d\tau}{d\kappa}\right)\dot{Q}$$

Spin Correction Factor $$\frac{d\tau}{d\kappa} = \frac{1}{\sqrt{1 - \frac{|\dot{a}|^2}{c^2\lambda^2}}} = \sqrt{1 + \frac{|\mathring{a}|^2}{c^2\lambda^2}}$$

② New Action Principle: Particles follow paths for which the length swept out by the momentum minus area swept out by spin is extremum

$$\mathcal{L} = m_0 c \sqrt{\mathring{x}^\mu \mathring{x}_\mu - \frac{1}{2\lambda^2}\mathring{a}^{\mu\nu}\mathring{a}_{\mu\nu}}$$

③ Canonical Momenta has Spin Corrected Mass

$$p_\mu = \frac{\delta \mathcal{L}}{\delta \mathring{x}^\mu} = m_0 \mathring{x}_\mu = m \dot{x}_\mu$$

$$S_{\mu\nu} = \lambda^2 \frac{\delta \mathcal{L}}{\delta \mathring{a}^{\mu\nu}} = m_0 \mathring{a}_{\mu\nu} = m \dot{a}_{\mu\nu}$$

$$m = m_0 \frac{d\tau}{d\kappa} = m_0 \sqrt{1 + \frac{|S|^2}{(m_0 c \lambda)^2}}$$

III.B Autoparallel Paths

① Parallel transport of basis vector

$$\frac{de_u}{dK} = \left(\dot{x}^\sigma \frac{\partial}{\partial x^\sigma} + \frac{1}{2}\dot{a}^{\alpha\beta}\frac{\partial}{\partial a^{\alpha\beta}}\right) e_u$$

where $\frac{\partial}{\partial x^\sigma}(p^v e_v) = e_v(\nabla_u p^v) = e_v(\Gamma_{u\sigma}{}^v p^\sigma + \partial_u p^v)$

Conjecture: Bivector Derivative is like a loop –

$$\frac{\partial}{\partial a^{\alpha\beta}}(p^v e_v) = e_v[\nabla_\alpha, \nabla_\beta]p^v = e_v(R_{\alpha\beta u}{}^v p^u - T_{\alpha\beta}{}^\sigma \nabla_\sigma p^v)$$

② Autoparallels are the Papapetrou Eqns

$$0 = \frac{d}{dK}(e_u p^u) = e_u\left[\dot{x}^\sigma \nabla_\sigma + \frac{1}{2}\dot{a}^{\alpha\beta}[\nabla_\alpha, \nabla_\beta] + \frac{\partial}{\partial K}\right]p^u$$

$$0 = \frac{d}{dK}(e_{uv} S^{uv}) = e_{uv}\left[\dot{x}^\sigma \nabla_\sigma + \frac{1}{2}\dot{a}^{\alpha\beta}[\nabla_\alpha, \nabla_\beta] + \frac{\partial}{\partial K}\right]S^{uv}$$

③ Terms in Red are Non Geodesic, Violate EEP

$$\dot{p}^u + p^v \dot{x}^\beta \Gamma_{\beta v}{}^u + \frac{1}{2}\dot{x}^\sigma S^{\alpha\beta} R_{\alpha\beta\sigma}{}^u = 0$$

$$\dot{S}^{\alpha\beta} + \dot{x}^v\left(S^{\delta\beta}\Gamma_{v\delta}{}^\alpha + S^{\alpha\delta}\Gamma_{v\delta}{}^\beta\right) + \frac{1}{2m}S^{uv}\left(S^{\delta\beta}R_{uv\delta}{}^\alpha + S^{\alpha\delta}R_{uv\delta}{}^\beta\right) = 0$$

III.C Curved Space Lagrangian

Spinning Particles follow paths which are NOT 4D geodesics but ARE autoparallels in 'Polygeometric space'

① Lagrangian - Just add metric!

$$\mathcal{L} = m_0 \sqrt{\mathring{x}^\mu \mathring{x}^\nu g_{\mu\nu} - \frac{1}{2\lambda^2} \mathring{a}^{\mu\alpha} \mathring{a}^{\nu\beta} g_{\mu\nu} g_{\alpha\beta}}$$

② Dirac Canonical Form (Better to quantize) vary p independent of x...

$$\mathcal{L} = p_\mu \mathring{x}^\mu - \frac{1}{2\lambda^2} S_{\mu\nu} \mathring{a}^{\mu\nu} - \frac{1}{2m_0}\left(p_\alpha p_\beta g^{\alpha\beta} - \frac{S_{\mu\alpha} S_{\nu\beta} g^{\mu\nu} g^{\alpha\beta}}{2\lambda^2}\right)$$

③ 3rd Form Vary δm_0 as Lagrange Multiplier

$$\mathcal{L} = m_0\left(\mathring{x}_\mu \mathring{x}^\mu - \frac{\mathring{a}_{\mu\nu} \mathring{a}^{\mu\nu}}{2\lambda^2}\right) + \frac{1}{m_0}\left(p_\mu p^\mu - \frac{S_{\mu\nu} S^{\mu\nu}}{2\lambda^2}\right)$$

④ Possible Extensions: add mass as canonical to scalar (+ Weyl connection)

$$\mathcal{L} = \sqrt{-\dot{\phi}^2 + \dot{x}^2 - \frac{1}{2\lambda^2} \dot{a}^2}$$

$$= -m_0 \dot{\phi} + p_\mu \dot{x}^\mu - \frac{1}{2\lambda^2} S_{\mu\nu} \dot{a}^{\mu\nu} - \Lambda\left(m_0^2 - p^2 + \frac{S^2}{2\lambda^2}\right)$$

multiplier

IV.A Anholonomic Mechanics

① Variation of Lagrangian

$$\delta \mathcal{L} = \left(\frac{\delta \mathcal{L}}{\delta x^\alpha}\delta x^\alpha + \frac{\delta \mathcal{L}}{\partial \dot{x}^\alpha}\delta \dot{x}^\alpha\right) + \frac{1}{2}\left(\frac{\delta \mathcal{L}}{\delta a^{\alpha\beta}}\delta a^{\alpha\beta} + \frac{\delta \mathcal{L}}{\delta \dot{a}^{\alpha\beta}}\delta \dot{a}^{\alpha\beta}\right)$$

$$p_\alpha \delta \dot{x}^\alpha = \underbrace{\cancel{\frac{d}{d\kappa}\left(p_\alpha \delta x^\alpha\right)}}_{\text{Fixed endpts}\ \nearrow 0} - \underbrace{\mathring{p}_\alpha \delta x^\alpha}_{\text{Standard}} + \underbrace{p_\alpha\left(\delta \dot{x}^\alpha - \frac{d}{d\kappa}\delta x^\alpha\right)}_{\text{Does Not vanish!}}$$

② Variation of Velocity component is Not the velocity of variation component

Shortcut: $\delta(dx^\mu e_\mu) = d(\delta x^\mu e_\mu)$

$$e_\alpha\left(\delta \dot{x}^\alpha - \frac{d}{d\kappa}\delta x^\alpha\right) = \left(\delta x^\alpha \mathring{e}_\alpha - \dot{x}^\alpha \delta e_\alpha\right)$$

$$= e_\sigma\left[\delta x^\alpha \dot{x}^\beta \tau_{\alpha\beta}{}^\sigma + \tfrac{1}{2}\left(\delta x^\alpha \mathring{a}^{\mu\nu} - \dot{x}^\alpha \delta a^{\mu\nu}\right) R'_{\mu\nu\alpha}{}^\sigma\right]$$

$$\left(\delta \dot{a}^{\alpha\beta} - \frac{d}{d\kappa}\delta a^{\alpha\beta}\right) = 2\left(\delta a^{\sigma\beta}\dot{x}^\mu - \mathring{a}^{\sigma\beta}\delta x^\mu\right)\Gamma_{\mu\sigma}{}^\alpha + \left(\delta a^{\sigma\beta}\mathring{a}^{\mu\nu} - \mathring{a}^{\sigma\beta}\delta a^{\mu\nu}\right)R'_{\mu\nu\sigma}{}^\alpha$$

Note $R'_{\mu\nu\sigma}{}^\alpha \equiv R_{\mu\nu\sigma}{}^\alpha - \tau_{\mu\nu}{}^\lambda \Gamma_{\lambda\sigma}{}^\alpha$

③ with this, get Papapetrou eqns!

$$\mathring{p}_\alpha - \frac{\delta \mathcal{L}}{\delta x^\alpha} = p_\sigma \tau_{\alpha\beta}{}^\sigma \dot{x}^\beta + p_\sigma R'_{\mu\nu\alpha}{}^\sigma \mathring{a}^{\mu\nu} + S_{\sigma\beta} \tau_{\alpha\mu}{}^\sigma \mathring{a}^{\mu\beta}$$

IV.B Hamiltonian Canonical Mechanics

useful for Quantizing!

① Legendre Transformation $\mathcal{H} = p_\alpha \mathring{x}^\alpha - \frac{1}{2\lambda^2} S_{\alpha\beta}\, \mathring{a}^{\alpha\beta} - \mathcal{L}$

$$\delta\mathcal{H} = \left(\frac{\partial\mathcal{H}}{\partial p_\alpha}\delta p_\alpha + \frac{\partial\mathcal{H}}{\partial x^\alpha}\delta x^\alpha\right) + \frac{1}{2}\left(\frac{\partial\mathcal{H}}{\partial S_{\alpha\beta}}\delta S_{\alpha\beta} + \frac{\delta\mathcal{H}}{\delta a^{\alpha\beta}}\delta a^{\alpha\beta}\right)$$

$$= \left(\mathring{x}^\alpha \delta p_\alpha - \frac{\partial\mathcal{L}}{\partial x^\alpha}\delta x^\alpha\right) - \frac{1}{2\lambda^2}\left(\mathring{a}^{\alpha\beta}\delta S_{\alpha\beta} + \frac{\partial\mathcal{L}}{\partial a^{\alpha\beta}}\delta a^{\alpha\beta}\right)$$

② Non commutivity of $[\delta, d] \neq 0$ destroys symmetry of Hamilton's Equations

ok $\mathring{x}^\mu = \frac{\delta\mathcal{H}}{\delta p_\mu}$ $\qquad \mathring{a} = -\lambda^2 \frac{\delta\mathcal{H}}{\delta S_{\mu\nu}}$

BUT

$$\frac{\delta\mathcal{H}}{\delta x^\lambda} = -\mathring{p}_\lambda + p_\sigma T_{\lambda\rho}{}^\sigma \frac{\delta\mathcal{H}}{\delta p_\rho} + p_\sigma R'_{\mu\nu\lambda}{}^\sigma \frac{\delta\mathcal{H}}{\delta S_{\mu\nu}} - S_{\sigma\beta}\Gamma_{\lambda\mu}{}^\sigma \frac{\delta\mathcal{H}}{\delta S_{\mu\nu}}$$

③ Poisson Brackets is $\{F, G\} \to \frac{1}{i\hbar}[\hat{F}, \hat{G}]$ valid?

$$\mathring{F} = \left(\frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial p}\dot{p}\right) + \frac{1}{2}\left(\frac{\partial F}{\partial a}\dot{a} + \frac{\partial F}{\partial S}\dot{S}\right)$$

$$= \underbrace{\{F, \mathcal{H}\}} + \text{Torsion + Curvature Terms}$$

$$\left(\frac{\partial F}{\partial x^\alpha}\frac{\partial\mathcal{H}}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial\mathcal{H}}{\partial x^\alpha}\right) - \frac{\lambda^2}{2}\left(\frac{\partial F}{\partial a^{\mu\nu}}\frac{\partial\mathcal{H}}{\partial S_{\mu\nu}} - \frac{\partial F}{\partial S_{\mu\nu}}\frac{\partial\mathcal{H}}{\partial a^{\mu\nu}}\right)$$

V. References

- W. Pezzaglia, Polydimensional Relativity, a Classical Generalization of the Automorphism Invariance Principle, in *Clifford Algebras and Applications in Mathematical Physics*, Dietrich (Kluwer 1998), gr-qc/9608052
- W. Pezzaglia, Physical Applications of a Generalized Clifford Calculus, in *Dirac Operators in Analysis* (Pitman Research Notes in Mathematics, Number 394), J. Ryan (Longman 1997) pp. 191-202, gr-qc/9710027
- W.G. Dixon, Dynamics of extended bodies in general relativity, I. Momentum and Angular Momentum, Proc. Roy. Soc. Lond. A314, 499(1970)
- A. Papapetrou, Spinning test-particles in general relativity, I, Proc. Roy. Soc. London A209, 248 (1951)
- P. Fiziev and H. Kleinert, New Action Principle for Classical Particle trajectories in Spaces with Torsion, Europhys. Lett. 35, 241 (1996), hep-th/9503074
- M. Pavsic, Clifford Algebra Based Polydimensional Relativity and Relativistic Dynamics, hep-th/0011216.
- W. Pezzaglia, Dimensionally Democratic Calculus and Principles of Polydimensional Physics, in *Proceedings of 5th Conference on Clifford Algebras and their applications in mathematical physics, Vol 1*, R. Ablamowicz & B. Fauser (1999), gr-qc/9912025
- W. Pezzaglia, New Classical Action Principle for Equations of Motion of Spinning Particles in Curved Space (The Papapetrou Equations), preprint http://www.clifford.org/wpezzag/preprints/pezz9801/