Spin-Gauge Theories

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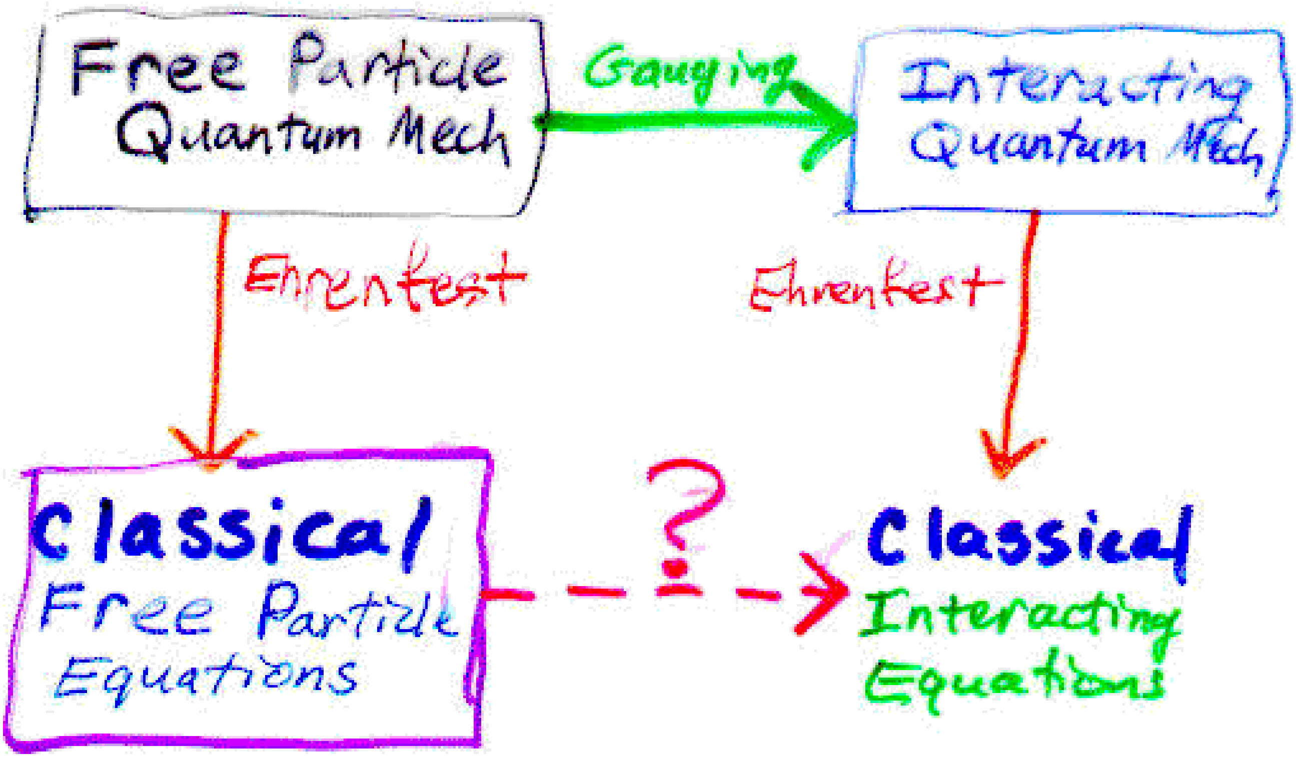
Physics should be independent of the particular representation chosen for spin (Pauli/Dirac) matrices. By gauging the spin group one can introduce interactions, n.b. a gauge theory of gravity. Realizing the spin matrices (a Clifford algebra) can be replaced by the spacetime geometric algebra, one is led to consider that all interactions can be interpreted as curvature which is trans- dimensional (e.g. a vector parallel transported around a closed loop may be meta-morphed into a scalar plus bivector-plane).

Cal Poly Pomona, July 2001, Lecture 3
http://www.clifford.org/~wpezzag/talks.html

July 3-6

I Introduction

Conventional Gauge theory exploits internal quantum symmetries to introduce interactions - even though some of these are classical



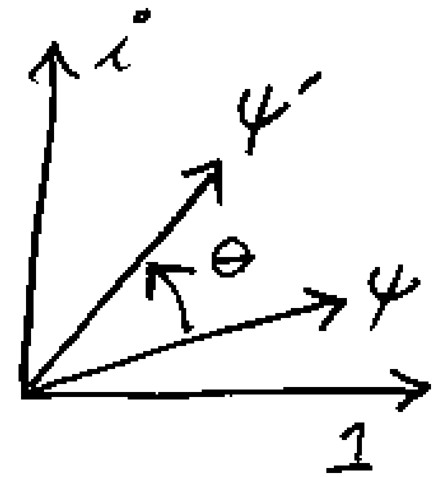
? where is the classical analogy of the gauge principle

II. Review of Conventional Theory

Need i as generator of electric charge

A. ~~Schrodinger Equation~~

1. Global U(1) Inv



- Global change in Quantum phase

$$\psi' = e^{i\theta}\psi$$

is an isometry of complex plane

$$\|\psi'\| = \|\psi\|$$

- Lagrangian and all observables (momentum + spin) are unchanged

$$\mathcal{L} = \frac{-\hbar^2}{2m}\partial_j\psi^*\partial_j\psi - i\hbar\left(\psi^*\partial_t\psi - \partial_t\psi^*\psi\right)$$

- Noether's Theorem gives conserved (probability) current

$$\partial_t(\psi^*\psi) = \vec{\nabla}\cdot\vec{J}$$

$$J_k = -\frac{i\hbar}{2m}\left(\psi^*\partial_k\psi - \partial_k\psi^*\psi\right)$$

II.A.② Local Gauge Transformation

- Postulate physics is "form invariant" (covariant) under position (path-history) dependent changes in phase.

$$\psi'(x) = U(x)\,\psi(x)$$

$$U(x) = \exp\left[\frac{ie}{\hbar c}\int_{\text{Path}}^{x} A_\mu(y)\,dy^\mu\right]$$

(Yang's Non-integrable phase factor – 1974)

- Define "Gauge Covariant Derivative"

$$\partial_\mu\left(U^{-1}\psi'\right) = U^{-1}\mathcal{D}_\mu\psi'$$

$$\mathcal{D}_\mu \equiv \partial_\mu - \frac{ie}{\hbar c}\underbrace{A_\mu}_{\text{Gauge Connection}}$$

- Curvature of gauge connection is the electromagnetic field tensor

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -\frac{ie}{\hbar c}F_{\mu\nu}$$

II.A.3 Equations of Motion

- Lagrangian: Substituting $\partial_\mu \to D_\mu$ generates interaction which is analogous to Classical electrodynamics

$$\mathcal{L}_I = J_\mu A^\mu \quad , \quad J_4 = e\psi^+\psi$$

$$J_j = e\frac{i\hbar}{2m}\left[(\psi^+ D_\mu^+)\psi - \psi^+(D_\mu\psi)\right] \simeq \frac{e}{m}p_j$$

- **Note:**

Mechanical Momenta: $p_j = \langle -i\hbar D_j \rangle$

Canonical Momenta: $k_j = p_j + A_j = \langle -i\hbar\partial_j \rangle$

Quantize $\quad [x_j, \hat{k}_m] = i\hbar\delta_{jm}$

- **Ehrenfest's Theorem**

$$\frac{dp}{dt} = \frac{1}{i\hbar}[\hat{p}, \mathcal{H}] + \left\langle \frac{\partial p}{\partial t} \right\rangle$$

After some messy steps recover classical:

$$\dot{\vec{p}} = e(\vec{E} + \vec{v}\times\vec{B}) \qquad \text{Lorentz Force}$$

$$\dot{\mathcal{E}} = e(\vec{v}\cdot\vec{E}) \qquad \text{Work-Energy}$$

II. B Pauli Equation

Concentrate on spin contribution

1. Lagrangian

(a) Free Particle: Replace $\partial_j \to \sigma_j \partial_j = \nabla$

$$(\psi^+ \overleftarrow{\partial_j})(\partial_j \psi) \to (\psi^+ \nabla)(\nabla \psi) = \partial_j \psi^+ \sigma_j \sigma_k \partial_k \psi$$

$$= \underbrace{(\partial_j \psi^+)(\partial_j \psi)}_{\text{Dot Term Schrodinger}} + \underbrace{(\partial_j \psi^+) \sigma_j \wedge \sigma_k \partial_k \psi)}_{\text{Wedge term - vanishes for free particle}}$$

(b) Interaction: $\partial_j \to D_j = \partial_j - \frac{ie}{\hbar c} A_j$

$$\mathcal{L} = -\frac{\hbar^2}{2m}(\psi^+ D_j^+) \sigma_j \sigma_k (D_k \psi) - \frac{i\hbar}{2}[\psi^+ D_t \psi - (\psi^+ D_t)\psi]$$

$$= \underbrace{\mathcal{L}_0 + J_\mu A^\mu}_{\text{Schrodinger}} + \underbrace{\mathcal{L}_s}_{\text{Spin}}$$

(c) Wedge term gives magnetic interaction

$$\mathcal{L}_s = (\psi^+ D_j^+) \sigma_j \wedge \sigma_k (D_k \psi) \frac{\hbar^2}{2m}$$

$$= \vec{A} \cdot (\vec{\nabla} \times \vec{S}) \mu_B \cong (\vec{B} \cdot \vec{S}) \mu_B$$

Spin current

Dipole in B Field

integrate by parts

II.B.(2) Equations of Motion

verify Pauli Equation describes particle with spin and magnetic moment.

(a) Pauli Equation has simple form using D_j

$$-\frac{\hbar^2}{2m}(\sigma_K D_K)^2 \psi = i\hbar D_t \psi$$

Gauge Covariant Derivative $D_\mu = \partial_\mu - \frac{ie}{\hbar c} A_\mu$

(b) Hamiltonian is Schrodinger + Spin Part

$$\mathcal{H} = \mathcal{H}_{schrod} + \underbrace{2}_{g\text{ factor}} \mu_B B^j \frac{\sigma_j}{2}$$

(c) Ehrenfest Theorem Recovers Classical Eqns

Force $\quad \dot{\vec{p}} = e(\vec{E} + \vec{V}\times\vec{B}) - \nabla(\vec{\mu}\cdot\vec{B})$

Work $\quad \dot{\mathcal{E}} = e(\vec{E}\cdot\vec{V}) + \frac{\partial}{\partial t}(\vec{\mu}\cdot\vec{B})$

Torque $\quad \dot{\vec{S}} = \vec{\mu}\times\vec{B}$

11.B③ How to get Spin-Orbit Coupling?

Although this is non-relativistic theory, the electromagnetic interaction - Lorentz Force - is relativistically correct.

Spin Contribution is NOT relativistically correct, missing velocity terms! Can get (nearly) correct by making replacement

$$\vec{B} \rightarrow \left(\vec{B} - \frac{\vec{v}}{c^2} \times \vec{E}\right)$$

$$\begin{cases} \mathcal{H}_I = \vec{\mu} \cdot \left(\vec{B} - \frac{\vec{v}}{c^2} \times \vec{E}\right) \\ \dot{\vec{p}} = e\left(\vec{E} + \vec{v} \times \vec{B}\right) - \nabla\left[\vec{\mu} \cdot \left(\vec{B} - \frac{\vec{v}}{c^2} \times \vec{E}\right)\right] \\ \dot{\vec{S}} = \mu \times \left(\vec{B} - \frac{\vec{v}}{c^2} \times \vec{E}\right) \end{cases}$$

* ? ⇒ How Get this from Pauli Equation from gauge principle? You Can't! Ad hoc -

$$\mathcal{L}_I = \frac{e\hbar^2}{8m^2c^2}\left[(\partial_j \psi^\dagger)\,\sigma_j \sigma_k \psi + \psi^\dagger \sigma_k \sigma_j \partial_j \psi\right] E^k$$

$$\mathcal{H} = \mathcal{H}_{pauli} + \frac{e\hbar^2}{8m^2c^2}\Big(\underbrace{\nabla E}_{\nabla \cdot E \,+\, \nabla \wedge E} - \underbrace{2E \wedge \nabla}_{\text{Thomas}}\Big)$$

Darwin — vanish for Central Field — SPIN Orbit

This ad-hoc approach gives Darwin and Thomas (Spin-orbit coupling) terms. But did not follow from Gauge Principle.

* [Pezzaglia Notes 7/29/93 unpublished]

II.C Relativistic Equations & Interaction

we'll begin to explore alternatives here

① Klein Gordon Equation

(a) Free Particle $(\square^2 - m^2)\phi = 0$

$$\square^2 \equiv \partial_\mu \partial^\mu = \square \cdot \square$$

$$\mathcal{L}_0 = \hbar^2 (\partial_\mu \phi^+)(\partial^\mu \phi) + m^2 \phi^+ \phi$$

(b) Interaction - Replace $\partial_\mu \rightarrow D_\mu = \partial_\mu - \frac{ie}{\hbar c} A_\mu$

$$\mathcal{L} = \hbar^2 (\phi^+ D_\mu^+)(D_\mu \phi) + m^2 \phi^+ \phi$$

$$= \mathcal{L}_0 + J_\mu A^\mu$$

Similar interaction as classical (and Pauli equation) except J_4 is not positive definite

Current $J_\mu = \frac{-ie\hbar}{2m}\left[\phi^+(D_\mu \phi) - (\phi^+ D_\mu^+)\phi\right]$

(c) Wave eqn: $(D^\mu D_\mu - m^2)\phi = 0$
expand -

$$(\square^2 - m^2 - A^2 - 2ie\hbar A \cdot \square)\phi = ie\hbar \underbrace{(\square \cdot A)}_{\text{set to zero}}\phi$$

II.C. (2) Feynmann - Gell-Mann Equation

Derive from Klein-Gordon in analogy to how Feynmann got Pauli from Schrodinger.

(a) Put geometry in: Replace dot with direct product.

$$\Box \cdot \Box \longrightarrow \Box\,\Box$$

$$\Box \equiv e^{\mu} \partial_{\mu}$$

$$(\phi^{+}\Box)(\Box\phi) = (\partial_{\mu}\phi^{+})\, e^{\mu} e^{\nu} (\partial_{\nu}\phi)$$

$$= \underbrace{(\partial_{\mu}\phi^{+}\partial^{\mu}\phi)}_{\substack{\text{Dot term} \\ \text{Klein-Gordon}}} + \underbrace{(\partial_{\mu}\phi^{+})\, e^{\mu}\wedge e^{\nu} (\partial_{\nu}\phi)}_{\substack{\text{wedge term} \\ \text{vanish for Free particle}}}$$

(b) For Interactions Replace ∂_{μ} with $\partial_{\mu} \rightarrow D_{\mu} = \partial_{\mu} - \frac{ie}{\hbar c} A_{\mu}$

$$\mathcal{L} = \hbar^{2} (\phi^{+} D_{\mu}^{+})\, e^{\mu} e^{\nu} (D_{\nu}\phi) + m^{2} \phi^{+}\phi$$

$$= \mathcal{L}_{0} + J_{\mu} A^{\mu} + \mathcal{L}_{\wedge}$$

$$\mathcal{L}_{\wedge} = \hbar^{2} (\phi D_{\mu}^{+})\, e^{\mu}\wedge e^{\nu} (D_{\nu}\phi)$$

$$= \frac{ie\hbar}{c} A_{\mu}\, \partial_{\nu} (\phi^{+} e^{\mu}\wedge e^{\nu} \phi)$$

$$= i \frac{\mu_{B}}{m} F_{\mu\nu} (\phi^{+} e^{\mu}\wedge e^{\nu} \phi)$$

integrate by parts

Hence magnetic moment must be $\mathcal{U}^{\mu\nu} \simeq \mu_{B}\, i\, (\phi^{+} e^{\mu}\wedge e^{\nu}\phi)$

II. C.2 (C) F.G.M. Wave Equation + Soln

In covariant form $\left(e^{\nu}\partial_{\nu}e^{\mu}\partial_{\mu}-m^{2}\right)\phi=0$

expand

$$\left(\Box^{2}-m^{2}-A^{2}-2ie\hbar A\cdot\Box\right)\phi=ie\hbar\left(\Box\cdot A+\underbrace{\Box\wedge A}_{\text{new Bivector term}}\right)\phi$$

Its obvious the solution ϕ MUST be multigeometric. There are TWO sectors of solutions, each with 8 parts

even $\phi=\phi_0 1+\frac{1}{2}\phi_2^{\mu\nu}e_\mu\wedge e_\nu+\phi_4\hat{e}$

odd $\phi=\phi_1^{\mu}e_\mu+\phi_3^{4}\hat{e}e_4$

where $\hat{e}=e_1e_2e_3e_4$

For Constant magnetic field in z direction

$$\phi(x)\simeq\underbrace{(1+ie_1\wedge e_2)}_{\text{spin Idempotent}}e^{ik\cdot x}$$

Note $\frac{ie\hbar}{c}B^{3}e_1\wedge e_2\,(1+ie_1\wedge e_2)=\underbrace{\frac{e\hbar}{c}B^{3}}_{\text{eigenvalue}}(1+ie_1\wedge e_2)$

More General $\phi\simeq(1+i\underline{S})e^{ik\cdot x}$

$$\underline{S}=\tfrac{1}{2}S^{\mu\nu}e_\mu\wedge e_\nu=\hat{e}\,s\wedge p/m$$

$s=s^{\nu}e_\nu$ Pauli-Lubanski Spin Polarization

II.C. (3) Dirac Equation

- Factor Feynmann-Gell-Mann Equation.

$$(e^\mu D_\mu e^\nu D_\nu - m^2)\phi = \left[(\square - \tfrac{ie}{\hbar c} A)^2 - m^2\right]\phi$$

$$= (\square - ieA - m)\underbrace{(\square - ieA + m)\phi}_{\text{multivector } \psi}$$

- Multivector Field ψ derived from potential ϕ satisfies the Dirac equation

$$(\square - ieA - m)\psi = 0$$

$$\psi = (\square - ieA + m)\phi$$

$$= (e^\mu D_\mu + m)\phi$$

- Momentum eigenstates: if $\phi \simeq e^{ik\cdot x}$

$$\psi \cong \underbrace{(1 + iP/m)}_{\text{idempotent}} e^{ik\cdot x}$$

Recall ϕ had spin structure, get

$$\psi_{p,s} \simeq \sqrt{\frac{m}{p^4}}\,(1 + iP/m)(1 + \hat{\epsilon}\,\delta)e^{ik\cdot x}$$

Note
$$(1+iP/m) = (1+iP/m)\,iP/m$$
$$(1+iP/m)(1+i\epsilon\,\delta \wedge P/m) = (1+iP/m)(1+\epsilon\,\delta)$$

Hence, particle at rest with z axis spin would have form

$$\psi \simeq (1+\underbrace{ie_4}_{\text{vector momentum}})(1+\underbrace{\hat{\epsilon}e_3}_{\text{trivector spin}})e^{-imt}$$

Greider exploited this "idempotent" structure to interpret the electron as a collection of properties of different geometries.

The conserved current $\partial_\mu(\bar{\psi}e^\mu\psi)=0$ Leads to multivector conserved quantities:

$$Q = \int d^3x\ \bar{\psi}e^4\psi$$

Assuming $\psi \simeq \sqrt{\frac{m}{p^4}}(1+ip/m)(1+\epsilon s)e^{ik\cdot x}$

$$\bar{\psi} = \sqrt{\frac{m}{p^4}}(1+ip/m)(1+\epsilon s)e^{-ik\cdot x}$$

$$Q \simeq \frac{m}{p^4}(1+\epsilon s)(1+ip/m)e^4(1+ip/m)(1+\epsilon s)$$

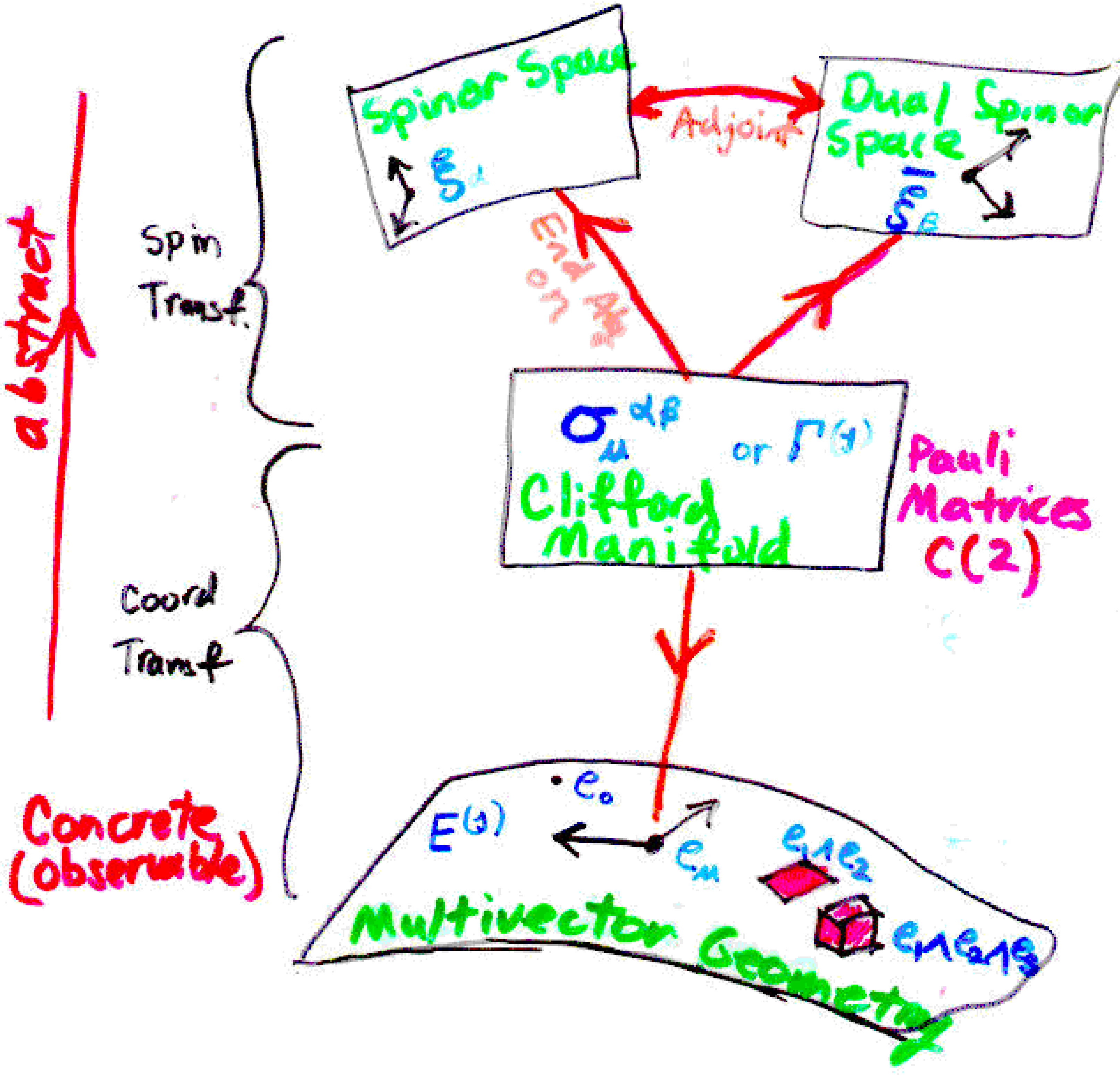
several steps

$$= i\ (1+ip/m)(1+\epsilon s)$$

$$= i + \frac{p}{m} + \hat{\epsilon}s\wedge\frac{p}{m} + i\hat{\epsilon}s$$

Charge "Imaginary" — momentum — Magnetic moment Bivector — Spin trivector

III Spin Gauge Theory



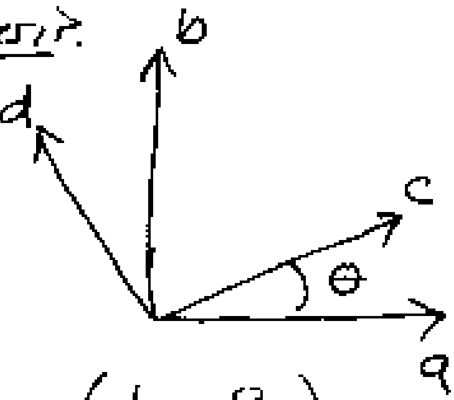
$$e_\mu \equiv \xi_\alpha \sigma_\mu^{\alpha\beta} \xi_\beta^{+}$$

III. A Spin Gauge on Pauli Eqn

Unpublished original work 1991-93
summarized in master thesis by Matt Engalran 95

(1.) Global Change in Spin Basis

A change in matrix representation of Pauli Matrizes is a global rotation of spin basis



$$\psi = \begin{pmatrix} a \\ b \end{pmatrix} \begin{matrix} z \text{ up} \\ z \text{ down} \end{matrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\psi' = \begin{pmatrix} c \\ d \end{pmatrix} \begin{matrix} x \text{ up} \\ x \text{ down} \end{matrix} \qquad \sigma_1' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Transformation is unitary SU(2)

$$\psi' = U\psi$$

$$\sigma_j' = U\sigma_j U^+$$

$$U(\theta_1, \theta_2, \theta_3) = \exp(i\sigma_j \theta^j/2)$$

Transformation is an Isometry of spin space C_2 (two complex component linear space). Preserves all quadratic forms.

Norm $\psi'^+\psi' = \psi^+\psi$

Spin $S_j' = \psi'^+\sigma_j'\psi' = \psi^+U^+U\sigma_j U^+U\psi$

$$= \psi^+\sigma_j\psi = S_j$$

III.A.1 (b) Conserved Current

The Lagrangian is invariant under this

$$\mathcal{L} = \frac{\hbar^2}{2m}(\partial_j \psi^+)\sigma_j\sigma_k(\partial_k \psi) - \frac{i\hbar}{2}(\psi^+\partial_t\psi - \partial_t\psi^+\psi)$$

Noether's theorem tells us there are conserved currents associated with symmetry

$$\partial_\mu B^\mu_{[j]} = 0 \qquad \text{for } j = 1, 2, 3$$

$$B_j^{\ \mu} = \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \psi}\frac{\delta \psi}{\delta \theta^j} - \frac{\delta \psi^+}{\delta \theta^j}\frac{\delta \mathcal{L}}{\delta \partial_\mu \psi^+}\right)$$

Conserved Quantities $\quad \frac{d}{dt}\int d^3x\, B^4_{[j]} = 0$

For our case $\quad \delta\psi = i\underline{\underline{\sigma_j}}\psi\,\delta\theta^j$, yields

Spin is Conserved $\quad B^4_{[j]} = \hbar\,\psi^+\underline{\underline{\sigma_j}}\psi$

Spin Flux Current

$$B^k_{[j]} = \frac{-i\hbar}{2m}(\psi^+\sigma_j\partial^k\psi - \partial^k\psi^+\sigma_j\psi)$$

$$\simeq S_j V^k$$

Note $\text{Trace}(B^k_{[j]}) = S\cdot V$ Helicity!

III.A. (2) Local Spin-Gauge Transformations

- Postulate physics is "form invariant" (Covariant) under position (path-history) dependent changes in spin basis (local changes in matrix representation)

$$\psi'(x) = U(x)\,\psi(x)$$

$$U(x) = \exp\left[\frac{\lambda}{\hbar c}\int_{\text{Path}}^{x} i\sigma_K W_\mu^{[K]}\, dy^\mu\right]$$

λ = coupling

- Spin-Gauge Covariant Derivative

$$\partial_\mu(U^{-1}\psi') = U^{-1} D_\mu \psi'$$

$$D_\mu \equiv \partial_\mu - \frac{e}{\hbar c}\, iA_\mu - \frac{\lambda}{\hbar c}\, i\sigma_K W_\mu^{[K]}$$

$$U(1) \otimes SU(2)$$

- You can treat E+M as 4th component of $W_\mu^{[0]}$ where $i\cdot\sigma_0 = i$. Curvature of full gauge connection is $U(1)\otimes SU(2)$ Yang Mills

$$[D_\mu, D_\nu] = \frac{-ie}{\hbar c} F_{\mu\nu} - \frac{\lambda}{\hbar c}\, i\sigma_K \mathcal{F}_{\mu\nu}^{[K]}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{F}_{\mu\nu}^{[K]} = \partial_\mu W_\nu^{[K]} - \partial_\nu W_\mu^{[K]} + \epsilon^{Kij} W_\mu^{[i]} W_\nu^{[j]} \frac{\lambda}{\hbar c}$$

III. A. (3) Wave equation

$$\mathcal{L} = \frac{-\hbar^2}{2m}(\psi^\dagger D_j^\dagger)\sigma_j\sigma_k(D_k\psi) - \frac{i\hbar}{2}\left(\psi^\dagger D_t\psi - (\psi^\dagger D_t^\dagger)\psi\right)$$

$$D_\mu = \partial_\mu - \frac{ie}{\hbar c}A_\mu - \lambda\frac{i\sigma_k}{\hbar c}W_\mu^{[k]}$$

(a) Decomposition of Gauge Connection
In electroweak Yang Mills $U(1)\otimes SU(2)$ the gauge group commutes with spin algebra. Here the gauge group is the spin algebra, which causes a collapse of ~~the~~ several degrees of freedom.

Starting with $W_\mu^{[k]}$, 12 components,

$$\sigma_j D_j = \vec{\nabla} - \frac{ie}{\hbar c}\vec{A} - \frac{\lambda}{\hbar c}\underbrace{\sigma_j i\sigma_k}_{i\delta_{jk} - \epsilon_{jkn}\sigma^n} W_j^{[k]}$$

We should rewrite this in terms of auxiliary potentials (gauge connection)

$$-i\hbar\sigma_j D_j = -i\hbar\nabla - \underset{\text{scalar}}{\lambda\phi} - (\underset{\text{vector}}{e\vec{A}} + \underset{\text{pseudovector}}{\lambda i\vec{C}})$$

Scalar $\phi = W_k^{[k]}$ $k = 1, 2, 3$

Pseudo vector $C^j = \epsilon^{jmn} W_n^{[m]}$

We have "lost" 5 degrees of freedom – the (traceless) symmetric part of W_μ^{ν}.

III.A.3 (b) Interaction terms

This is sketchy — but roughly the Lagrangian will have terms like:

$$\lambda \hbar\, W_4^{\,K}\, \psi^\dagger \sigma_K \psi \qquad \text{couple to spin}$$

$$+ \left(\vec{C}^{\,2} + \phi^2\right) \psi^\dagger \psi \qquad \text{Scalar interaction}$$

$$+ \; C \cdot \nabla \left(\psi^\dagger \psi\right) \qquad \text{Boundary term} \to 0$$

$$+ \; i\phi \left(\partial_j \psi^\dagger \sigma_j \psi - \psi^\dagger \sigma_j \partial_j \psi\right) \qquad \text{Helicity coupling to Scalar } \phi$$

$$+ \, i C^K \epsilon_{ijK} \left(\partial_i \psi^\dagger \sigma_K \psi - \psi^\dagger \sigma_K \partial_j \psi\right) \qquad \text{coupling to spin current}$$

(c) Wave equation:

In terms of covariant derivative is trivially the same as Pauli — If we re-express in terms of Auxiliary potentials it will get very messy —

$$\frac{-\hbar^2}{2m} \left(\nabla - i\phi - \vec{C} - i\vec{A}\right)^2 = i\hbar\, \partial_t \psi + \left(A_4 + \sigma_K W_4^{\,K}\right) \psi$$

? What are corresponding Classical equations?

- One looks very much like thomas term, if make $\vec{C} \simeq \vec{E}$ electric field
- Probably we have terms that mimic spin coupling to gravity.

$$\frac{d\vec{S}}{dt} \simeq \frac{\phi}{2m}\, \vec{S} \times \vec{p} + \vec{S} \otimes \vec{W}_4^{\,K}$$

Lense Thirring?

III.B Dirac Equation and Spin Gauge

Ref Crawford, Chisholm + Farwell

1. Global Symmetry

The Dirac Bispinor has 4 complex degrees of freedom, a "vector" in linear space $C^{2,2}$ meaning metric $(+ + - -)$

$$\psi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

Norm $\quad \bar{\psi}\psi = |a|^2 + |b|^2 - |c|^2 - |d|^2$

$$= \bar{\psi}^B \psi^A \eta_{AB} \leftarrow \text{"Spin Metric"}$$

(a) Isometry group is $U(2,2)$ [I think]

$$\psi' = U\psi$$

$$\gamma_\mu' = U \gamma_\mu U^{-1}$$

To preserve $\bar{\psi}\psi$ need $\bar{U} = U^{-1}$

$$U = \exp(\Gamma_A \, \theta^A/2)$$

hence "generators" obey $\bar{\Gamma}_A = -\Gamma_A$

In Dirac algebra $C(4)$ Γ_A could be,

$$\Gamma_A \in \{ \underset{1}{i}, \underset{4}{\gamma_\mu}, \underset{6}{\gamma_{\mu\nu}}, \underset{4}{\gamma_5\gamma_\mu}, \underset{1}{\gamma_5} \}$$

where $i = \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5$

Total of 16 generators (Pili group?)

(b) Conserved Currents

Dirac Lagrangian is invariant under this

$$\mathcal{L} = \frac{\hbar}{2}\left[(\partial_\mu \bar{\Psi})\gamma^\mu \Psi - \bar{\Psi}\gamma^\mu(\partial_\mu \Psi)\right] + mc\,\bar{\Psi}\Psi$$

Noether's theorem gives currents –

$$\partial_\mu B^\mu_{[A]} = 0$$

$$B^\mu_{[A]} = \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \Psi}\frac{\delta \Psi}{\delta \theta_A} - \frac{\delta \bar{\Psi}}{\delta \theta_A}\frac{\delta \mathcal{L}}{\delta \partial_\mu \bar{\Psi}}\right)$$

$$\delta \Psi = \frac{\Gamma_A}{2}\Psi\,\delta\theta^A$$

$$\boxed{B^\mu_{[A]} \cong \bar{\Psi}\{\gamma^\mu, \Gamma_A\}\Psi}$$

Γ_A	$B^\mu_{(A)}$	Interpretation
i	$i\bar{\Psi}\gamma^\mu\Psi$	Electric current j^μ
γ_5	$\emptyset$	No magnetic current?
γ_ν	$\bar{\Psi}\Psi g^{\mu\nu}$	? Norm Preserved? $\bar{\Psi}\Psi = 1$
$\gamma_5\gamma_\mu$	$\bar{\Psi}\gamma^5\gamma^{\mu\nu}\Psi$	magnetic moment
$\gamma_{\alpha\beta}$	$\bar{\Psi}\gamma^{\mu\alpha\beta}\Psi$	Spin

III.B.(2) Local Spin-Gauge Transformation

Propose physics is covariant under local transformations. Spin Covariant Derivative:

$$\mathcal{D}_\mu \equiv \partial_\mu - \Gamma_A \mathcal{W}_\mu^{[A]}$$

$$i A_\mu + \gamma_\nu \mathcal{M}_\mu{}^\nu + \tfrac{1}{2}\gamma_{\alpha\beta}\Omega_\mu^{\alpha\beta} + \gamma_5 \gamma_\nu F_\mu{}^\nu + \gamma_5 \mathcal{M}_\mu^5$$

E+M | ? Frame Field | Fock Ivanenko | ? | Pseudo vector

Insert this into Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \tfrac{1}{2}\bar{\Psi}\{\gamma^\mu, \Gamma_A\}\Psi\, \mathcal{W}_\mu^{[A]}$$

The symmetrization with γ^μ kills some terms.

Γ_A	Interaction	Interpretation
i	$(i\bar{\Psi}\gamma^\mu\psi)A_\mu$	E+M
$\gamma_5\gamma_\nu$	$(\bar{\Psi}\gamma^5\gamma^{\sigma\mu}\psi)F_\mu{}^\sigma$	Anomalous Magnetic Moment?
γ_5	$\emptyset$	$\mathcal{M}_\mu^5$ Does not couple
$\gamma_{\alpha\beta}$	$(\bar{\Psi}\gamma^{\alpha\beta\mu}\psi)\Omega_\mu^{\alpha\beta}$	Couples to Spin
γ_ν	$\bar{\Psi}\Psi\ \mathrm{Trace}(\mathcal{M}_\mu{}^\mu)$	<u>Mass</u> term (Higgs?)
		Rest of $\mathcal{M}_\mu{}^\nu$ does not couple!

III.B.③ Wave Equation

$$(\gamma^\mu D_\mu - m)\psi = 0$$

$$D_\mu = \partial_\mu - iA_\mu + \gamma_\nu M_\mu{}^\nu + \gamma_{\alpha\beta}\Omega_\mu^{\alpha\beta} + \gamma_5\gamma_\nu F_\mu{}^\nu + \gamma_5 M_\mu^5$$

Contraction with γ^μ decomposes, kills some terms and repartitions them.

$$\gamma^\mu D_\mu = \square \underbrace{- i A^\mu \gamma_\mu}_{4 \text{ E+M}} + \underbrace{M_\mu{}^\mu 1}_{\text{Mass! } 1}$$

$$+ \underbrace{\gamma_5 F_\mu{}^\mu}_{1 \text{ Pseudoscalar}} + (\underbrace{\gamma^\mu \cdot \gamma_{\alpha\beta}}_{4 \text{ vector}} + \underbrace{\gamma^\mu \wedge \gamma_{\alpha\beta}}_{+4 \text{ Trivector}})\Omega_\mu^{\alpha\beta} \quad \text{Fock Ivanenko}$$

$$+ \underbrace{\gamma_{\mu\nu}}_{\text{Bivector}} (\underbrace{M_\mu{}^\nu}_{+6} + \underbrace{\gamma_5 F_\mu{}^\nu}_{+6}) + \underbrace{\gamma_5 \gamma^\mu}_{\text{Pseudo vector}} \underbrace{M_\mu{}^5}_{+4}$$

Out of $4 \times 16 = 64$ components in D_μ, only 30 remain.

> Question: What Equations of motion does Ehrenfest's theorem give us for this complete connection?

Need to do!

- Should give Papapetrou (from Fock-Ivanenko) plus "New" terms which couple motion + spin to these fields.

III.C Spin Curvature

① Spin Basis

Conventional Dirac equation is spin basis dependent $\quad (\gamma^{\mu}_{AB}\partial_{\mu} - m\delta_{AB})\psi^{B} = 0$

Define Spin Basis (twistor), to make spin basis independent ψ

$$\underbrace{\psi}_{\text{manifest spin basis independent}} = \underbrace{\psi^{B}}_{\text{Spinor component}} \underbrace{\hat{\xi}_{B}}_{\text{Basis}}$$

Define Spin Metric $\quad \bar{\xi}_{A}\xi_{B} = \eta_{AB}$

$$\eta_{AB} = \begin{pmatrix} + & & & \\ & + & & \\ & & - & \\ & & & - \end{pmatrix}$$

The norm easily recovered $\quad \bar{\psi}\psi = \psi^{*A}\bar{\varepsilon}_{A}\varepsilon_{B}\psi^{B}$

$$= \psi^{*A}\psi^{B}\eta_{AB}$$

To make $\psi^{+}\psi$ = positive definite need to define

$$\xi^{A+}\varepsilon_{B} = \delta^{A}{}_{B}$$

$$\bar{\varepsilon}_{B} \equiv \varepsilon^{A+}\eta_{AB}$$

Can hence construct a spin basis (matrix representation) independent Clifford Algebra.

$$e_{\mu} \equiv \varepsilon_{B}\gamma_{\mu}{}^{B}{}_{A}\varepsilon^{A+}$$

$$e_{\mu}e_{\nu} = \varepsilon_{A}\gamma_{\mu}{}^{A}{}_{B}\varepsilon^{B+}\varepsilon_{C}\gamma_{\nu}{}^{C}{}_{D}\varepsilon^{D+} = \varepsilon_{A}\gamma_{\mu\nu}{}^{A}{}_{D}\varepsilon^{D+}$$

III.C. (2) Spin Connection

Propose spin basis is position-dependent, and has non-trivial "spin connection"

$$\partial_\mu \mathcal{E}_A = \mathcal{E}_B \Omega_\mu{}^B{}_A$$
$$= \big(\underbrace{\mathcal{E}_B \Omega_\mu{}^B{}_C \mathcal{E}^{C+}}_{\text{some collection of CA.}}\big)\mathcal{E}_A$$

$$\partial_\mu \mathcal{E}_A = \Omega_\mu \mathcal{E}_A$$
$$\Omega_\mu = \Omega_\mu{}^{[A]} E_{[A]}$$

Multivector Connection

$E_0 = 1,\ E_1 = e_1 \ \ldots$

Spin Covariant Derivative defines:

$$\partial_\mu (\psi^B \mathcal{E}_B) = (\partial_\mu \psi^B)\mathcal{E}_B + \psi^B(\partial_\mu \mathcal{E}_B)$$
$$= \mathcal{E}_B (D_\mu \psi^B)$$

$$D_\mu \equiv \partial_\mu + \Omega_\mu{}^{[A]} E_{[A]}$$

So the "Dirac Equation" is now written as spin covariant form:

$$0 = (\square - m)\psi = (e^\mu \partial_\mu - m)(\psi^B \mathcal{E}_B)$$
$$= \left[(e^\mu D_\mu - m)\psi^B\right]\mathcal{E}_B$$

III. C. (3) Spin Curvature

$$[\partial_\mu, \partial_\nu] \mathcal{E}_A = \mathcal{E}_B \underbrace{K_{\mu\nu}}_{\text{Space}}{}^{B} \underbrace{{}_A}_{\text{Spin}}$$

$$= \left(\mathcal{E}_B K_{\mu\nu}{}^{B}{}_{C}\, \mathcal{E}^{+C}\right)\left(\mathcal{E}_A\right) = \underline{K}_{\mu\nu} \mathcal{E}_A$$

$$\boxed{\underset{\text{Spin Curvature}}{\underline{K}_{\mu\nu}} \equiv K_{\mu\nu}{}^{[J]} E_{[J]} \quad \text{sum over multivector}}$$

In terms of Spin Covariant Derivative

$$[\partial_\mu, \partial_\nu] \psi^B \mathcal{E}_B \equiv \left([D_\mu, D_\nu] \psi^B\right) \mathcal{E}_B$$

$$= \underline{K}_{\mu\nu} \mathcal{E}_B \psi^B$$

The "problem" is to try to relate this spin curvature to space curvature.
If we accept the map: $e_\mu = \mathcal{E}_A \gamma_\mu{}^A{}_B \mathcal{E}^{B+}$

Then we can "derive" relationships from Riemann to Spin Curvature:

$$[\partial_\mu, \partial_\nu] e_\alpha = R_{\mu\nu\alpha}{}^{B} e_B$$

$$= K_{\mu\nu} e_\alpha + e_\alpha K_{\mu\nu}^{+}$$

$$+ \mathcal{E}_A \Big(\underbrace{[\partial_\mu, \partial_\nu]}_{\text{vanish?}} \gamma_\alpha{}^A{}_B\Big) \mathcal{E}^{B+}$$

In general, we find that we must restrict $K_{\mu\nu}$ or else vectors will be curved into bivectors etc.
Need $K_{\mu\nu}^{+} = -K_{\mu\nu}$ and bivector to preserve rank!

IV Multivector QM

- Throw away conventional spin space. Spinors can be represented by minimal left ideals of Clifford Algebra

$$\psi = \psi^A E_A$$

$$E_A \in \{1,\ e_\mu,\ e_{\mu\nu},\ \epsilon e_\mu,\ \epsilon\}$$

- Multivector ψ can contain multiple generations, isospin is in right ideal structure — multiplication on Right side of ψ will couple these.

$$\psi = \underbrace{\psi^B}_{\text{Dirac Bispinor}} \underbrace{\phi_A}_{\text{Isospinor}} \mathcal{E}_B\, \mathcal{E}^{+A}$$

- Dirac equation is manifestly coordinate free — valid for curved space.

$$(e^\mu \partial_\mu - m)\psi^A E_A = 0$$

where $e^\mu(x)$, $E_A(x)$ are functions of x.

- No "spin curvature" as no spin space. Can recover this only if allow vectors to rotate into bivectors etc.

IV.A Dextral Gauge Theory

Interactions applied to right side of ψ couple the generations of particles.

(1.) Symmetry Group

Lagrangian in general is no longer a scalar. Take scalar part

$$\mathcal{L} = \langle (\bar{\psi}\Box)\psi - \bar{\psi}(\Box\psi) - 2m\bar{\psi}\psi \rangle_0$$

"Norm" $Sp(\bar{\psi}\psi) = Sp(\psi\bar{\psi}) = \langle \bar{\psi}\psi \rangle_0$

(a) Isometry	$\psi' = U\psi$	$\psi' = \psi S$
Applied	Left side	Right side
Vector Rule	$e_\mu' = U e_\mu U^{-1}$	$e_\mu' = e_\mu$
Transformation	Spin	Isospin

Either U or S have same structure

$$U \cong \exp(\Gamma_A\, \theta^A/2)$$

$$\bar{\Gamma}_A = -\Gamma_A \quad \text{for} \quad \bar{U} = U^{-1}$$

$$\Gamma_A \in \{ i,\ e_\mu,\ e_{\mu\nu},\ e_5 e_\mu,\ e_5 \}$$

The "i" is shared by both. Total Group

$$U(1) \times \underset{\text{Left side}}{SO(6)} \times \underset{\text{Right side}}{SO(6)}$$

(b) Conserved Currents.

Dirac Lagrangian - if we take scalar part is invariant under this transformation on the right side

$$\delta\Psi = \Psi \frac{\Gamma_A}{2}\delta\theta^A , \quad \delta\bar{\Psi} = -\frac{\Gamma_A}{2}\bar{\Psi}\delta\theta^A$$

$$\mathcal{L} = \frac{\hbar}{2}\left[(\partial_\mu\bar{\Psi})e^\mu\Psi - \bar{\Psi}e^\mu(\partial_\mu\Psi)\right] + mc\bar{\Psi}\Psi$$

Noether's theorem gives current for each Γ_A

$$B^\mu_{[A]} = \left\langle \frac{1}{2}\left\{ \bar{\Psi}e^\mu\Psi, \Gamma_A \right\} \right\rangle_0$$

These are hence the individual parts of Greiner's multivector current!

$$\bar{\Psi}e^\mu\Psi = ij^\mu + T^{\mu\nu}e_\nu + \mathcal{U}^{\mu\alpha\beta}e_{\alpha\beta} + \mathcal{W}^{\mu\nu}e_5e_\nu + J^\mu e_5$$

$$= B^\mu_A \Gamma^A$$

So we've finally been able to "prove" Greiner's ~~postulated~~ empirically constructed multivector current from a formal method

IV. A (2) Local Dextrad Gauge Transf

Propose physics is covariant under local Dextrad transformations. But how is this really justified? We no longer can
* blame it on 'spin space' or matrix representation. Discuss later.

$$\partial_\mu(\psi' S^{-1}) = \left(\partial_\mu \psi' + \psi'(\partial_\mu S^{-1}) S\right) S^{-1}$$
$$= \left(\partial_\mu \psi' + \psi' \Omega_\mu\right) S^{-1}$$

$$\Omega_\mu = \Omega_\mu{}^A E_A = \Omega_{1\mu}{}^\nu e_\nu + \tfrac{1}{2}\Omega_{2\mu}{}^{\alpha\beta} e_{\alpha\beta} + \cdots$$

Similar to spin-gauge connection, its a multivector, except its applied to the right of wave function, so we can. NOT pull it thru and define a gauge covariant derivative.

In Lagrangian or wave equation we instead replace $\Box\psi$ by

$$\Box\psi + e^\mu \psi e_\nu \Omega_{1\mu}{}^\nu + e^\mu \psi e_{\alpha\beta} \Omega_{2\mu}{}^{\alpha\beta} + \cdots$$
$$= \Box\psi + e^\mu \psi E_A \Omega_\mu{}^{[A]}$$

So interaction term of Lagrangian is just

$$\mathcal{L}_I = \langle \bar{\psi} e^\mu \psi, E_A \rangle_0 \Omega_\mu{}^{[A]}$$

$$= B^\mu{}_{[A]} \Omega_\mu{}^{[A]}$$

Currents couple to connection as desired!

Can define curvature -

$$[\partial_\mu, \partial_\nu](\psi' S^{-1}) = (\psi' E_A K_{\mu\nu}{}^A) S^{-1}$$

$$K_{\mu\nu}{}^A E_A = \partial_\mu(\Omega_\nu{}^B E_B) - \partial_\nu(\Omega_\mu{}^B E_B)$$

③ <u>Wave equation & Current Conservation</u>

$$\Box \psi = m\psi + e^\mu \psi E_A \Omega_\mu{}^A$$

Multiply on left by $\bar{\psi}$, and then construct Bar adjoint

$$-(\bar{\psi}\Box)\psi = m\bar{\psi}\psi + E_A \bar{\psi} e^\mu \psi \Omega_\mu{}^A$$

Subtract to get multivector conservation law -

$$\partial_\mu(\bar{\psi} e^\mu \psi) = [(\bar{\psi} e^\mu \psi), E_A]\Omega^A{}_\mu$$

OR, $\partial_\mu (B^\mu_{[A]} E^A) = B^\mu_{[D]} [E_D, E_A] \Omega_\mu{}^{[A]}$

This is basically the Yang-Mills type -

$$D_\mu J = 0 \Rightarrow \partial_\mu J^\mu + J \times A = 0$$

Note $\partial_\mu E^A \neq 0$ in general!

IV.B Poly Covariant Theory

The problem that I was faced with:

- We have thrown away spin space, hence can not invoke "matrix representation" or "spin basis" covariance to justify introducing spin gauge fields.
- Our wave functions are now multivectors of the space-time geometry
- Interactions of form $E_A \Psi E_B$ "bilateral" in general are what we desire to recover all the wonderful physics we saw in "standard" spin gauge theory.
- These bilateral forms algebraically are general linear maps which reshuffle geometry.
- We must therefore propose the principle that physics is invariant under local 'polydimensional' transformations
- This idea will recover Crawford's spin gauge, plus the Dextral connection, plus new "bilateral" interactions that you can't derive from either of these.

IV. B. (1) Geobeins

- At any point in space, can always find a local Clifford algebra frame which satisfies $\{e_\mu, e_\nu\} = 2g_{\mu\nu}$.

- A short distance away, the local set of geometric objects can be resolved onto this fiducial set by the analogy of a tetrad field – "Geobein"

$$E'_A(x) = \hat{E}_B \, \Delta^B{}_A(x) \longleftarrow \text{geometric Legs}$$

So e'_μ may be part vector + bivector relative to fiducial set.

- If you force $\{E'_A\}$ to also be a Clifford algebra with $\{e'_\mu, e'_\nu\} = 2g_{\mu\nu}$ ~~then~~ i.e. obey exactly the same algebraic rules, then the geobeins describe an automorphism transformation (i.e. equivalent to change in matrix representation)

- In general, the E'_A might not correspond to an automorphism on $\hat{E}_B$. It may be an Endomorphism.

IV.B (2) Polydimensional Connection

Now we have connection that do not preserve the rank of geometry

$$\partial_\mu e_\alpha = \Gamma_{\mu\alpha}{}^{\nu} e_\nu + \Xi_{\mu\alpha}{}^{\beta\sigma} e_{\beta\sigma} + \cdots$$

We are more interested in how $\Box$ behaves operating on any element in the algebra. Define poly connection

$$\Box E_{(i)} = \Lambda_{(i)}{}^{(j)} E_{(j)}$$

So $\Lambda_i{}^j$ is the endomorphism algebra which maps the 16 elements into 16 elements (for 4D space).

Polydimensional wave equation:

$$0 = (\Box - m)(\psi^A E_A) \qquad \text{manifestly poly-invariant}$$

$$= \left[(\Box - m)\psi^A + \psi^B \Lambda_B{}^A\right] E_A$$

The problem is that each particular ~~part~~ component of ψ^A has a different connection. The structure is in terms of multivector object, rather than in terms of the left-right ideal structure.

Can rewrite $\Lambda_i{}^j E_j = E_A E_j E_B \Omega^{AB}$ as a 2 sided "bilateral" product. Note that this form is independent of particular element E_j, hence can rewrite connection on full ψ

$$(\Box - m)\psi + E_A \psi E_B \Omega^{AB} = 0$$

IV.B(3) <u>Currents</u>

Back up a bit — Show that Lagrangian is invariant under polydim transformation which leave $\langle \bar{\Psi} \Psi \rangle_0 = \langle \Psi \bar{\Psi} \rangle_0$ invariant

This will restrict transformation:

$$\delta \Psi = E_A \Psi E_B \; \delta \Theta^{AB} \qquad \begin{matrix} A, B \text{ go} \\ 1 \text{ to } 16 \end{matrix}$$

$$\delta \bar{\Psi} = \bar{E}_B \bar{\Psi} \bar{E}_A \; \delta \Theta^{AB}$$

may be trouble which defn of "Bar" not preserved under some transformations . . .

$$\langle \delta \bar{\Psi} \Psi = - \bar{\Psi} \delta \Psi \rangle_0 \Rightarrow \langle \bar{E}_B \bar{\Psi} \bar{E}_A \Psi - \bar{\Psi} E_A \Psi E_B \rangle_0 = 0$$

will be satisfied if ~~both~~ E_B and E_A are <u>Both</u> bar negative or both bar positive only!

Noether's Thm gives "Conserved Current:"

$$\partial_\mu \langle \bar{\Psi} e^\mu E_A \Psi E_B + E_B \bar{\Psi} E_A e^\mu \Psi \rangle_0 = 0$$

For interacting equation - these currents couple to the bilateral field Ω^{AB}

$$\mathcal{L}_I = \langle \bar{\Psi} e^\mu E_A \Psi E_B + E_B \bar{\Psi} E_A e^\mu \Psi \rangle \Omega^{AB}$$

* Conservation law for current with Ω^{AB} will be messy —

IV. C <u>Multivector Field Eqn</u>

we come full circle here

Generalized Curvature $\quad [\partial_\mu, \partial_\nu] E_A = \mathcal{K}_{\mu\nu A}{}^B E_B$

we'd like to rewrite so works for collection of element, so rewrite in bilinear form –

$$[\partial_\mu, \partial_\nu]\, Q = \mathcal{K}'_{\mu\nu}{}^{AB} E_A Q E_B$$

For example:

$$[\partial_\mu, \partial_\nu]\, e_{\sigma} = \tfrac{1}{2} R_{\mu\nu}{}^{\alpha\beta} \left(e_{\alpha\beta} e_\sigma - \cancel{1} e_\sigma e_{\alpha\beta} \right)$$

Field eqn: For Yang Mills looks like ~

$$K^{\mu\nu} \equiv \mathcal{K}^{\mu\nu}{}_i E^i$$

$$K_{\mu\nu}{}^{;\mu} + [\Omega^\mu, K_{\mu\nu}] = J_\mu$$

to write this eqn in CA. without any indices, need a 2nd commuting CA - Eddington's EF numbers.

$$K \simeq \underbrace{\gamma_{\mu\nu}}\, K^{\mu\nu}{}_i\, \underbrace{E^i}$$

I suspect the Field eqn can have simple form

$$\Box K = J$$

all the messy stuff above will appear naturally from $\Box$ operating on γ_μ and E^i.

Generalized ~~[illegible]~~ —

we still have been simplistic in only having coordinates for vector —

$$\mathcal{L} = \bar{\psi}(\Box\psi) - (\bar{\psi}\Box)\psi + \ldots$$

Consider having Full Dimensional Democracy — so that replace

$$\Box\psi \rightarrow e^{\mu}\partial_{\mu}\psi + e^{\mu\nu}\frac{\partial}{\partial q^{\mu\nu}}\psi + \cdots$$

~~[illegible]~~

Consider gravitation could be perhaps

$$\mathcal{L} \simeq (e_{\mu}\Box)(\Box e^{\mu})$$

or a background field

$$\mathcal{L} = (\bar{E}_A \overleftarrow{D})(D E_A)$$

every geometry

Full derivative

— END —