



*Department of Physics*

# Classical Spin Electrodynamics in Background Curvature and Torsion

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- Lagrangian for Electrodynamics with Classical Spin
- Derivation of Covariant Stern-Gerlach Force
- New(?) EM Gauge dependent coupling to Tidal Gravity

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<http://www.clifford.org/~wpezzag/talks.html>

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Talk Saturday March 30, 10:00-10:12 am

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# I. Covariant Stern-Gerlach Force

Standard formulation does not generate correct covariant form from the Lagrangian?

## A. Gradient Force on Electric Dipole

$$\vec{F} = (\vec{\rho} \cdot \nabla) \vec{E} \cong \nabla(\vec{\rho} \cdot \vec{E})$$

## B. Relativistic Form

Dipole Tensor:

$$\mathcal{U}^{\alpha\beta} = \frac{ge}{2m} S^{\alpha\beta}$$

Lagrangian:

$$L_I = \frac{1}{2} \mathcal{U}^{\alpha\beta} F_{\alpha\beta} = \vec{\rho} \cdot \vec{E} + \vec{\mu} \cdot \vec{B}$$

Force:

$$f_\mu = \frac{\delta L}{\delta x^\mu} = \frac{1}{2} \partial_\mu (\mathcal{U}^{\alpha\beta} F_{\alpha\beta})$$

(if  $\partial_{[\alpha} F_{\beta\sigma]} = 0$ )

$$= \mathcal{U}^{\alpha\beta} \partial_\beta F_{\alpha\mu}$$

## C. Covariant Form

Several authors Dixon[1], van Holten[2], Chou[3] argue the covariant form is instead,

$$f_\mu = \mathcal{U}^{\alpha\beta} \nabla_\beta F_{\alpha\mu} = \mathcal{U}^{\alpha\beta} \left( \partial_\beta F_{\alpha\mu} + \Gamma_{\beta\mu}^\sigma F_{\sigma\alpha} - \Gamma_{\alpha\beta}^\sigma F_{\sigma\mu} \right)$$

## II. Dynamic Mass

### A. Stern-Gerlach $\Rightarrow$ NonConserved Mass

Eqn of Motion:  $\dot{p}_\sigma = e\dot{x}^\nu F_{\nu\sigma} + \frac{1}{2}\partial_\sigma (\mathcal{U}^{\alpha\beta} F_{\alpha\beta})$

Contract with  $\dot{x}$ :  $\dot{m} = -\dot{x}^\sigma \dot{p}_\sigma = -\frac{d}{d\tau} (\mathcal{U} \cdot F)$

### B. Barut's Theory

Lagrangian:  $L = m\sqrt{\dot{x}^\mu \dot{x}_\mu} + e\dot{x}^\mu A_\mu$

Dynamic Mass:  $m = m_0 - \frac{1}{2}\mathcal{U}^{\alpha\beta} F_{\alpha\beta}$

Ad-Hoc Spin Eq:  $\dot{S}_{\mu\nu} = (A \wedge j)_{\mu\nu} + (\mathcal{U} \otimes F)_{\mu\nu}$

### C. Features

- Spin magnitude is conserved
- Gauge-Dependent Torque (unphysical?)
- Curved Space Equation of Motion shows dynamic mass is both the inertial and gravitational mass :

$$(m_0 - \frac{1}{2}\mathcal{U}^{\alpha\beta} F_{\alpha\beta}) (\ddot{x}^\sigma + \dot{x}^\alpha \dot{x}^\beta \Gamma_{\alpha\beta}^\sigma)$$

$$= e\dot{x}^\nu F_{\nu\sigma} + \frac{1}{2}(g^{\sigma\mu} - \dot{x}^\sigma \dot{x}^\mu) \partial_\sigma (\mathcal{U}^{\alpha\beta} F_{\alpha\beta})$$

Does it make sense that Inertial mass can go **ZERO** or **NEGATIVE**?

### III. New Gradient Torque

#### A. Dixon's Analysis of Extended Bodies

Invariant: 
$$m_o^2 = p_\mu p^\mu - \frac{1}{2\lambda^2} S^{\mu\nu} S_{\mu\nu}$$

Spin Enhanced Mass: 
$$m = m_o \sqrt{1 + \left( \frac{\|S\|}{m_o c \lambda} \right)^2}$$

#### B. PolyDimensional Transformation

Rotating lines into planes preserves Invariant

$$\delta p_\sigma = \lambda^{-1} S_{\sigma\mu} \delta \phi^\mu, \quad \delta S_{\alpha\beta} = \lambda (p \wedge \delta \phi)_{\alpha\beta}$$

#### C. Metamorphic Covariance

Propose Laws of Physics Invariant under infinitesimal polydimensional transformations. Apply to momentum equation to derive spin equation of motion with **new gradient torque**.

Momentum: 
$$\dot{p}_\sigma = \frac{e}{m} \left[ p^\nu F_{\nu\sigma} + \frac{1}{2} S^{\alpha\beta} \partial_\sigma F_{\alpha\beta} \right]$$

Spin: 
$$\dot{S}_{\alpha\beta} = \frac{e}{m} \left[ (S \otimes F)_{\alpha\beta} + \lambda^2 p^\sigma \partial_\sigma F_{\alpha\beta} \right]$$

Mass: 
$$m\dot{m} = p_\mu \dot{p}^\mu = \frac{1}{2\lambda^2} S_{\mu\nu} \dot{S}^{\mu\nu} = \frac{e}{2} S_{\mu\nu} \dot{F}^{\mu\nu}$$

## IV. Anholonomic Mechanics

### A. Modified Euler-Lagrange Equation

Variation: 
$$\delta L = \frac{\delta L}{\delta q} \delta q + k \delta \dot{q}, \quad k \equiv \frac{\delta L}{\delta \dot{q}}$$

Rewrite: 
$$k \delta \dot{q} = \frac{d}{d\tau} (k \delta q) - \dot{k} \delta q + k \left( \delta \dot{q} - \frac{d}{d\tau} \delta q \right)$$

E.L. Eqn: 
$$\left( \dot{k} - \frac{\delta L}{\delta q} \right) \delta q = k \left( \delta \dot{q} - \frac{d}{d\tau} \delta q \right)$$

### B. Variation of Velocity $\neq$ Velocity of Variation

Example of 3D Rotation: 
$$\delta \omega^k - \frac{d}{dt} \delta \theta^k = (\vec{\omega} \times \delta \vec{\theta})^k$$

Yields Euler's Equation: 
$$\dot{J}_k - \frac{\delta L}{\delta \theta^k} = (\vec{\omega} \times \vec{J})_k$$

### C. Relativistic Spin Equation

Lagrangian: 
$$L = L_o + \frac{1}{2} \mathcal{U}^{\mu\nu} F_{\mu\nu}$$

Dipole Tensor: 
$$\mathcal{U}_{\mu\nu} = \frac{g}{2} e \lambda^2 \dot{\theta}_{\mu\nu}$$

Canonical Spin: 
$$\mathcal{J}_{\mu\nu} = \frac{\delta L}{\delta \dot{\theta}^{\mu\nu}} = S_{\mu\nu} + \frac{g}{2} e \lambda^2 F_{\mu\nu}$$

- Substituting the Canonical Spin into relativistic Euler's equation recovers our spin equation
- Magnitude of Canonical Spin (NOT mechanical spin) is conserved.

# V. Polydimensional Action

## A. Dimensional Democracy

- Space is made of points, lines, planes ...
- Each geometric element has a coordinate.
- Event:  $d\Sigma = dx^\mu e_\mu + \frac{1}{2} da^{\mu\nu} e_\mu \wedge e_\nu + \dots$
- New Affine Parameter (based on Invariant)

$$d\kappa^2 = dx^\mu dx_\mu - \frac{1}{2\lambda^2} da^{\mu\nu} da_{\mu\nu}$$

## B. Action Principle

Particles follow paths (polygeodesics) which extremize the length swept out by the momentum  $\pm$  the area swept out by the spin.

$$L = m_o \sqrt{\overset{\circ}{x}^\mu \overset{\circ}{x}_\mu - \frac{1}{2\lambda^2} \overset{\circ}{a}^{\mu\nu} \overset{\circ}{a}_{\mu\nu}} \quad \overset{\circ}{Q} \equiv \frac{dQ}{d\kappa}$$

## C. Momenta has Spin Enhanced Mass

Momentum:  $p_\mu \equiv \frac{\delta L}{\delta \overset{\circ}{x}^\mu} = m_o \overset{\circ}{x}_\mu = m \dot{x}_\mu$

Spin:  $S_{\mu\nu} \equiv \lambda^2 \frac{\delta L}{\delta \overset{\circ}{a}^{\mu\nu}} = m_o \overset{\circ}{a}_{\mu\nu} = m \dot{a}_{\mu\nu}$

Mass:  $m = m_o \frac{d\tau}{d\kappa} = \sqrt{p^\mu p_\mu} = m_o \sqrt{1 + \left(\frac{S}{m_o \lambda c}\right)^2}$

## VI. Flat Space Electrodynamics

### A. One Lagrangian for Both Equations

$$L = m_o \sqrt{\dot{x}^\mu \dot{x}_\mu - \frac{1}{2\lambda^2} \dot{a}^{\mu\nu} \dot{a}_{\mu\nu}} + e \left( \dot{x}^\mu A_\mu - \frac{1}{2} \dot{a}^{\mu\nu} F_{\mu\nu} \right)$$

Current: 
$$j^\sigma = e \dot{x}^\sigma = \left( \frac{e}{m_o} \right) p^\sigma$$

Dipole: 
$$\mathcal{U}^{\mu\nu} = e \dot{a}^{\mu\nu} = \left( \frac{e}{m_o} \right) S^{\mu\nu}$$

### B. In terms of New Affine Parameter

$$\dot{p}_\sigma = j^\nu F_{\nu\sigma} + \frac{1}{2} \mathcal{U}^{\mu\nu} \partial_\sigma F_{\mu\nu}$$

$$\dot{S}_{\mu\nu} = (\mathcal{U} \otimes F)_{\mu\nu} + \frac{e\lambda^2}{2} \dot{x}^\sigma \partial_\sigma F_{\mu\nu}$$

### C. In terms of Proper Time

$$m\ddot{x}^\sigma = e\dot{x}_\nu F^{\nu\sigma} + \frac{1}{2} \frac{e}{m} S^{\mu\nu} [g^{\alpha\sigma} - \dot{x}^\alpha \dot{x}^\sigma] \partial_\alpha F_{\mu\nu}$$

$$\dot{S}_{\mu\nu} = \left( \frac{e}{m} \right) (S \otimes F)_{\mu\nu} + \frac{e\lambda^2}{2} \dot{x}^\sigma \partial_\sigma F_{\mu\nu}$$

### D. Unify Equations with Clifford Algebra

Polymomenta: 
$$\mathcal{M} \equiv p^\mu \mathbf{e}_\mu + \frac{1}{2\lambda^2} S^{\mu\nu} \mathbf{e}_\mu \wedge \mathbf{e}_\nu$$

$$\frac{m_o}{e} \dot{\mathcal{M}} = [\mathcal{M}, F] + \frac{1}{4} \{ \square, \{\mathcal{M}, F\} \} \quad , \quad \square \equiv \mathbf{e}^\mu \partial_\mu$$

## VII. Curved Space Mechanics

### A. Anholonomy Couples S and P

$$\left( \delta \overset{\circ}{x}^\sigma - \frac{d}{d\kappa} \delta x^\sigma \right) =$$

$$\delta x^\alpha \overset{\circ}{x}^\beta \tau_{\alpha\beta}^\sigma + \frac{1}{2} \left( \delta x^\alpha \overset{\circ}{a}^{\mu\nu} - \overset{\circ}{x}^\alpha \delta a^{\mu\nu} \right) R'_{\mu\nu\alpha}{}^\sigma$$

$$\left( \delta \overset{\circ}{a}^{\lambda\sigma} - \frac{d}{d\kappa} \delta a^{\lambda\sigma} \right) = \delta_{\omega\nu}^{\lambda\sigma} \left[ \Gamma_{\alpha\mu}^\omega \left( \overset{\circ}{x}^\alpha \delta a^{\mu\nu} - \overset{\circ}{a}^{\mu\nu} \delta x^\alpha \right) \right.$$

$$\left. + \frac{1}{4} R'_{\alpha\beta\mu}{}^\omega \left( \overset{\circ}{a}^{\mu\nu} \delta a^{\alpha\beta} - \overset{\circ}{a}^{\alpha\beta} \delta a^{\mu\nu} \right) \right]$$

$$R'_{\alpha\beta\mu}{}^\omega \equiv R_{\alpha\beta\mu}{}^\omega - \tau_{\alpha\beta}^\sigma \Gamma_{\sigma\mu}^\omega$$

### B. Generalized Equation of Motion

$$\overset{\circ}{k}_\mu - \frac{\delta L}{\delta x^\mu} = k_\sigma \overset{\circ}{x}^\alpha \tau_{\alpha\mu}^\sigma + \frac{1}{2} \left( k_\omega R'_{\alpha\beta\mu}{}^\omega - \frac{1}{\lambda^2} \mathcal{J}_{\omega\beta} \Gamma_{\mu\alpha}^\omega \right) \overset{\circ}{a}^{\alpha\beta}$$

$$\left( \overset{\circ}{J}_{\mu\nu} - \lambda^2 \tau_{\mu\nu}^\sigma \frac{\delta L}{\delta x^\sigma} \right) = \overset{\circ}{x}^\alpha \Gamma_{\alpha\mu}^\omega \mathcal{J}_{\omega\nu} + \frac{\lambda^2}{2} R'_{\mu\nu\sigma\omega} (\overset{\circ}{x} \wedge k)^{\sigma\omega}$$

$$+ \frac{1}{2} \left( \overset{\circ}{a} \otimes \mathcal{J} \right)_{\alpha\beta} \left[ \frac{1}{\lambda^2} \delta_{\mu\nu}^{\alpha\beta} + R'_{\mu\nu}{}^{\alpha\beta} \right] - \frac{1}{2} \overset{\circ}{a}^{\alpha\beta} R'_{\alpha\beta\mu}{}^\omega \mathcal{J}_{\omega\nu}$$

Canonical Momentum:

$$k_\sigma = p_\sigma + e A_\sigma$$

Canonical Spin:

$$\mathcal{J}_{\mu\nu} = S_{\mu\nu} + e \lambda^2 F_{\mu\nu}$$

## VIII. New Electrodynamic Terms

### A. Spin Equation

All New Torque terms proportional to  $\lambda^2$

$$\begin{aligned} \frac{1}{\lambda^2} \left( \overset{\circ}{S} - \mathcal{U} \otimes F \right)_{\mu\nu} &\simeq \frac{1}{2} (\mathcal{U} \otimes F + j \wedge A)^{\sigma\omega} R'_{\mu\nu\sigma\omega} \\ &+ \left( j^\alpha \Gamma_{\alpha\mu}^\omega + \frac{1}{2} \mathcal{U}^{\alpha\beta} R'_{\alpha\beta\mu}{}^\omega \right) F_{\omega\nu} \\ &+ \dots \text{torsion terms } \dots \end{aligned}$$

$$R'_{\alpha\beta\mu}{}^\omega \equiv R_{\alpha\beta\mu}{}^\omega - \tau_{\alpha\beta}^\sigma \Gamma_{\sigma\mu}^\omega$$

### B. Momentum Equation

- Makes Stern-Gerlach Fully Covariant
- New Gauge-Dependent Force Term:

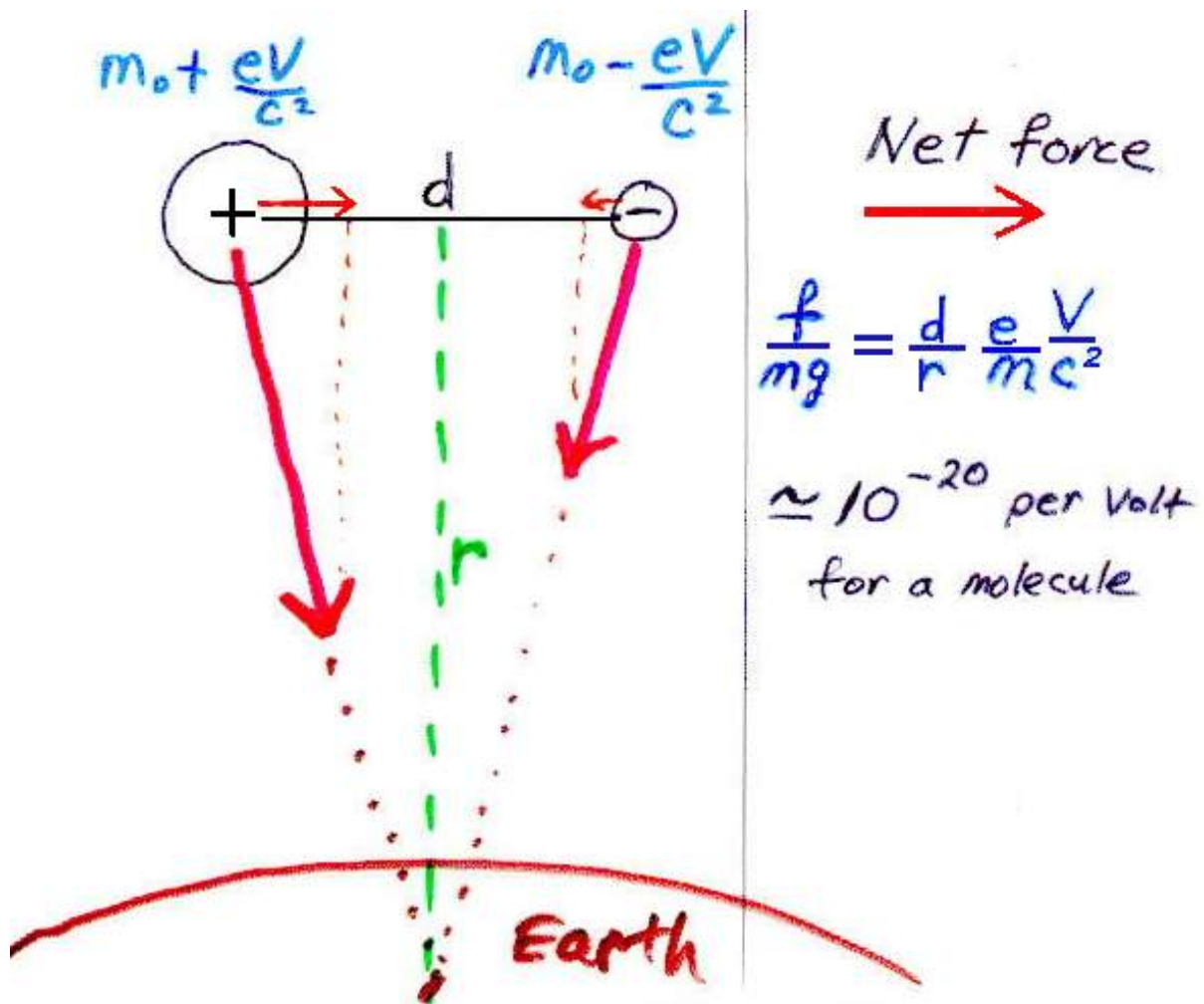
$$f_\mu = \frac{1}{2} \mathcal{U}^{\alpha\beta} \left( R_{\alpha\beta\mu}{}^\omega + \tau_{\alpha\beta}^\sigma \Gamma_{\sigma\mu}^\omega \right) A^\omega$$

- The force on a static electric dipole,

$$\vec{F} = A_4 \vec{\rho} \cdot \nabla \vec{g} = A_4 \nabla (\vec{\rho} \cdot \vec{g})$$

## IX. Interpretation

- Electrostatic PE adds to gravitational mass
- Electric Dipole in electrostatic potential acquires effective mass dipole of strength  $\sim A_4 \vec{p} / c^2$



## X. References

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2. **J. W. van Holten**, *On The Electrodynamics Of Spinning Particles*, Nucl. Phys. B356, 3 (1991).
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4. **Barut**, *Electrodynamics and Classical Theory of Fields and Particles*, (Dover, 1980), p. 73-80.
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