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If Particles follow Autoparallels Must then EM couple to Torsion?

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- Assume photons feel torsion
- Problems with conservation of magnetic flux
- Modification of conservation of charge

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`http://www.clifford.org/~wpezzag/talks.html`

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Talk Fri 2003Feb28 11:30-11:45 am

Mechanics with Torsion

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Lagrangian:}\\ \text{Path Length}\\ L = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\end{array}} \arrow[r, "\text{Turn On Torsion}"] \arrow[d, "\begin{array}{c}\text{Euler-}\\ \text{Lagrange}\\ \text{Eqns}\end{array}"] & \boxed{\begin{array}{c}\text{Lagrangian:}\\ \text{Same Form!}\\ L = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\end{array}} \arrow[d, swap, "\begin{array}{c}\text{Modified}\\ \text{Euler-}\\ \text{Lagrange}\\ {[d,\delta] \neq 0}\end{array}"] \\
\boxed{\begin{array}{c}\text{Geodesics:}\\ \ddot{x}^{\sigma} = -\{{}^{\sigma}_{\mu\nu}\}\dot{x}^{\mu}\dot{x}^{\nu}\end{array}} \arrow[r, "\text{Turn On Torsion}"] & \boxed{\begin{array}{c}\text{Autoparallels}\\ \ddot{x}^{\sigma} = -\Gamma^{\sigma}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\end{array}}
\end{tikzcd}$$

- Variation of velocity $\neq$ velocity of variation
- Massless particles follow null autoparallels?
- Photon feels torsion?

I. Anholonomic Mechanics

A. Particles in Torsion follow Autoparallels

$$\ddot{x}^\sigma = -\left[\left\{{}^{\sigma}_{\mu\nu}\right\} + \tfrac{1}{2}\left(\tau_\mu{}^\sigma{}_\nu + \tau_\nu{}^\sigma{}_\mu - \tau^\sigma{}_{\mu\nu}\right)\right]\dot{x}^\mu\dot{x}^\nu$$

Christoffel: $\left\{{}^{\sigma}_{\mu\nu}\right\} = \tfrac{1}{2}g^{\sigma\lambda}\left(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right)$

Lagrangian gives geodesics: $L = \sqrt{\dot{x}^\mu\,\dot{x}^\mu\,g_{\mu\nu}(x)}$

B. Modified Euler-Lagrange Equation

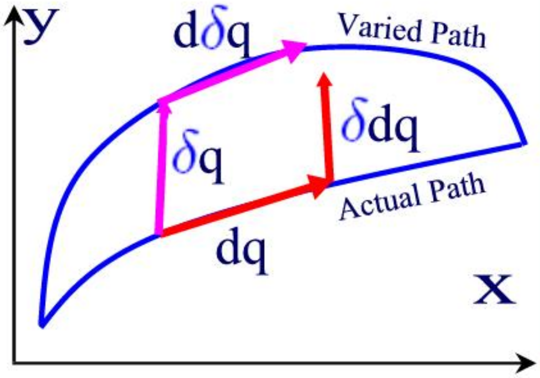
Variation: $\delta L = \frac{\delta L}{\delta q}\delta q + k\delta\dot{q}\ , \qquad k \equiv \frac{\delta L}{\delta\dot{q}}$

Rewrite: $k\,\delta\dot{q} = \frac{d}{d\tau}(k\delta q) - \dot{k}\delta q + k\left(\delta\dot{q} - \frac{d}{d\tau}\delta q\right)$

E.L. Eqn: $\left(\dot{k} - \frac{\delta L}{\delta q}\right)\delta q = k\left(\delta\dot{q} - \frac{d}{d\tau}\delta q\right)$

C. Variation of Velocity ≠ Velocity of Variation

Path of variations is not variation of the path?



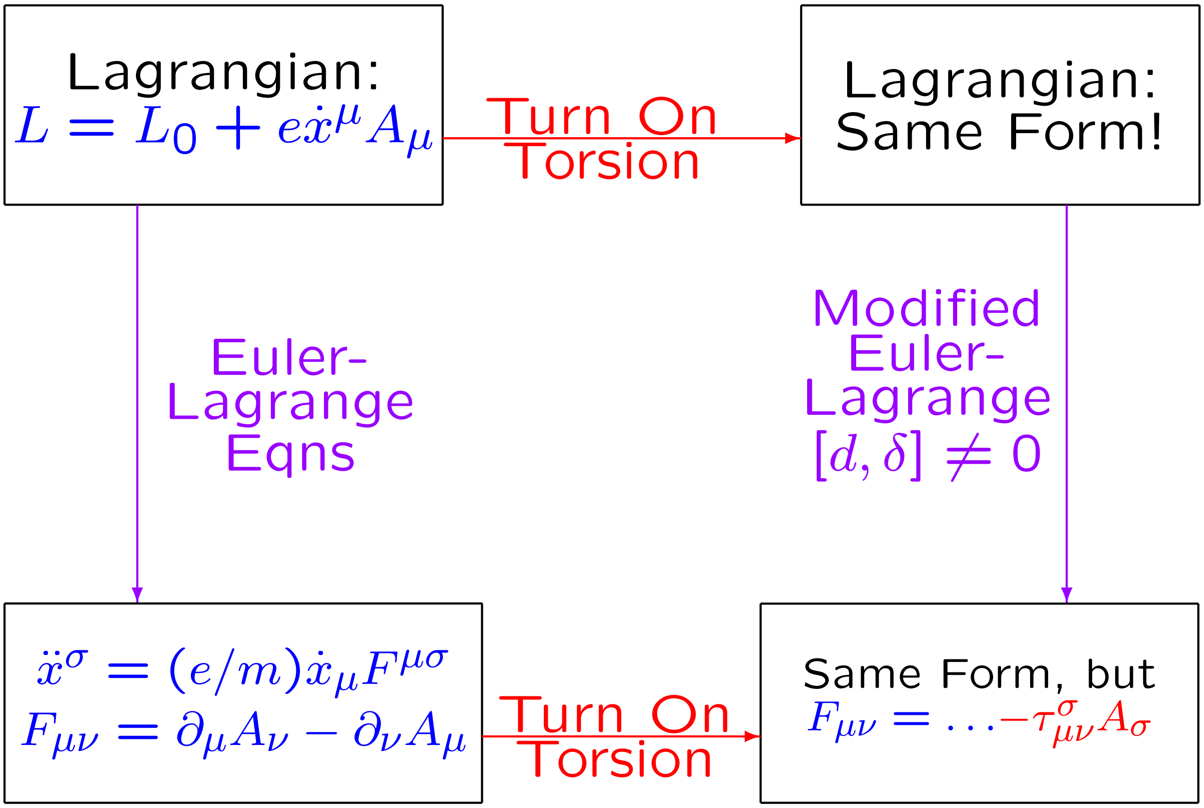
Require:
$\delta(dq^\sigma \mathbf{e}_\sigma) = d(\delta q^\sigma \mathbf{e}_\sigma)$

Derive:
$\delta\dot{q}^\sigma - \frac{d}{d\tau}\delta q^\sigma \quad = \quad \tau^\sigma_{\mu\nu}\dot{q}^\mu\delta q^\nu$

Yields autoparallels![1,2]

(even photons follow autoparallels?)

Electrodynamics and Torsion



Lagrangian:
$L = L_0 + e\dot{x}^{\mu} A_{\mu}$

Turn On Torsion

Lagrangian:
Same Form!

Euler-Lagrange Eqns

Modified Euler-Lagrange
$[d, \delta] \neq 0$

$\ddot{x}^{\sigma} = (e/m)\dot{x}_{\mu} F^{\mu\sigma}$
$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$

Turn On Torsion

Same Form, but
$F_{\mu\nu} = \ldots - \tau^{\sigma}_{\mu\nu} A_{\sigma}$

Torsion Introduces
Gauge-Dependent
Electrodynamic Force

$$\ddot{x}_{\mu} = \ldots - \frac{e}{m}\dot{x}^{\nu} \tau^{\sigma}_{\mu\nu} A_{\sigma}$$

II. Electrodynamics

Assume Interaction Lagrangian: $L_I = j^\mu A_\mu$
Euler-Lagrange Equations modified for torsion gives usual form,

$$\ddot{x}^\sigma + \dot{x}^\mu \dot{x}^\nu \Gamma^\sigma_{\mu\nu} = (e/m)\dot{x}_\mu F^{\mu\sigma}$$

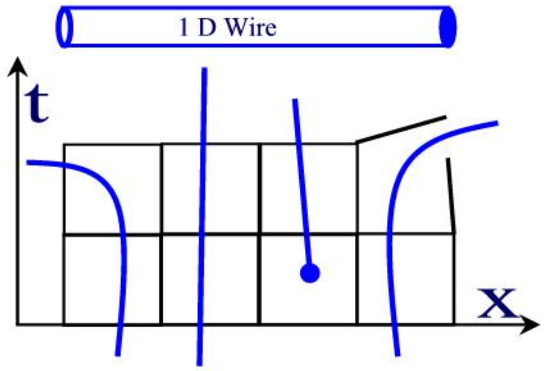
However, the field includes torsion (gauge dependent force):

$$F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]} \;=\; 2\partial_{[\mu}A_{\nu]} - \tau^\sigma_{\mu\nu}A_\sigma$$

Phenomenologically Torsion makes a charge "mimic" magnetic dipole and visa versa.

Source:	Charge e	Dipole μ
Potential	$A^4 = e/r$	$\vec{A} = \vec{\mu} \times \vec{r}/r^3$
xtra Field	$B_3 = \tau^4_{12} A_4$	$E_1 = \tau^k_{41} A_k$
Mimics	$\mathcal{U}_{12} = \tau^4_{12}\, e$	$e = \tau^1_{42} \mathcal{U}^{12}$
?	$\mathcal{U}_{\mu\nu} \simeq j_\sigma \tau^\sigma_{\mu\nu}$	$j_\sigma \simeq \tau^\mu_{\sigma\nu} \mathcal{U}_\mu{}^\nu$

III. Conservation of Charge



Torsion is a measure of NON-CLOSURE which can result in unaccounted flux leakage

Charge conservation, accounting for torsion,

$$0 = \oint_{\partial v} d^3\Sigma^\mu \sqrt{-g}\, j_\mu = \int_v d^4x \sqrt{-g}\, \nabla_\mu j^\mu$$

$$\nabla_\mu j^\mu = \partial_\mu \left(\ln(\sqrt{-g})\, j^\mu\right) + \tau^{\mu}_{\mu\sigma}\, j^\sigma = 0$$

Derivation from Gauge Principle: $\delta A_\mu = \delta \nabla_\mu \phi$

$$0 = \delta \int d^4x\, \mathcal{L}_I = \int_v d^4x \sqrt{-g}\, j^\mu \delta A_\mu$$

Integrate by parts, hold $\delta\phi = 0$ on boundary,

$$0 = \oint_{\partial v} d^3\Sigma^\mu \sqrt{-g}\, (j_\mu \delta\phi) = \int_v d^4x \sqrt{-g}\, \nabla_\mu (j^\mu \delta\phi)$$

Yields conservation law:

$$0 = \delta \int \mathcal{L}_I = -\int_v d^4x \sqrt{-g}\, \left(\delta\phi \nabla_\mu j^\mu + j^\mu [\nabla_\mu, \delta]\phi\right)$$

If commutator vanishes we get: $\nabla_\mu j^\mu = 0$

We *could* get $\partial_\mu(\sqrt{-g}\, j^\mu) = 0,$

If argue: $[\partial_\mu, \delta]\phi = -\tau^{\sigma}_{\sigma\mu}\delta\phi$

IV. Maxwell Equations

Metric-Free form of Homogeneous law is conservation of magnetic flux (Hehl[5]).

$$0 = \oint d\sigma^{\mu\nu} F_{\mu\nu} = \int d^3\Sigma_\mu \nabla_\nu F_{\sigma\omega} \epsilon^{\mu\nu\sigma\omega}$$

$$0 = \nabla_{[\sigma} F_{\mu\nu]} = \partial_{[\sigma} F_{\mu\nu]} - F_{\omega[\sigma} \tau^{\omega}_{\mu\nu]}$$

This is incompatible with: $F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]}$

$$0 \neq \nabla_{[\sigma} \nabla_\mu A_{\nu]}$$

$$= \left(\nabla_{[\sigma} \tau^{\omega}_{\mu\nu} - \tau^{\omega}_{\kappa[\sigma} \tau^{\kappa}_{\mu\nu]} \right) A_\omega - \left(\nabla_\omega A_{[\sigma} \right) \tau^{\omega}_{\mu\nu]}$$

We can

- Allow Magnetic Monopoles
- Ignore the Gauge Principle in QM
- Consider Dipole *Fluxoid* Sources $\mathcal{U}$

$$\boxed{\mathbf{dF = d}\mathcal{U}}$$

$$\nabla_{[\sigma} F_{\mu\nu]} = \nabla_{[\sigma} \nabla_\mu A_{\nu]} = \nabla_{[\sigma} \mathcal{U}_{\mu\nu]}$$

Equivalently, that fundamental sources have point charge **and** point dipole moment.

V. Inhomogeneous Equation

Hehl[5] argues that inhomogeneous Maxwell equation cannot be influenced by gravity.

$$\mathbf{j} \;=\; {}^*\mathbf{d}\,{}^*\mathbf{F}$$

$$j^\kappa = \partial_\mu \left[\ln\left(\sqrt{-g}\right) F^{\mu\kappa}\right] + \tau^\sigma_{\sigma\mu}\, F^{\mu\kappa} + \tfrac{1}{2}\tau^\kappa_{\mu\nu}\, F^{\mu\nu}$$

This is incompatible with charge conservation:

$$\nabla_\nu j^\nu = \nabla_{[\nu}\nabla_{\mu]}F^{\mu\nu} \;=\; -(R_{\mu\nu} - \tau^\sigma_{\mu\nu}\nabla_\sigma)F^{\mu\nu} \;\neq 0$$

<u>We can</u>

- Say EM doesn't couple to torsion,
- Assert $F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]}$ and hope some Bianchi identity will save us (sorry, it doesn't)
- Again consider Dipole *Fluxoid* Sources $\mathcal{U}$

$$\boxed{{}^*\mathbf{d}\,{}^*\mathbf{F} \;=\; \mathbf{j} \;+\; {}^*\mathbf{d}\,{}^*\mathcal{U}}$$

$$\nabla^\mu\nabla_{[\mu}A_{\nu]} \;=\; j_\nu + \nabla_\mu \mathcal{U}^\mu{}_\nu$$

We can hope that the dipole "spin" current restores conservation,

$$\nabla^\nu\nabla^\mu(\nabla_{[\mu}A_{\nu]} - \mathcal{U}_{\mu\nu}) \;=\; \nabla_\sigma j^\sigma \;=\; 0\ ?$$

VI. Derivation from Lagrangian

Define: $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - \tau^\sigma_{\mu\nu} A_\sigma$

$$\mathcal{A} = \int_v d^4x \sqrt{-g}\, \left(\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + j^\sigma A_\sigma - \tfrac{1}{2} F_{\mu\nu} \mathcal{U}^{\mu\nu} \right)$$

Variation: $\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta A_\nu} \delta A_\nu + \Pi^{\mu\nu} \delta(\partial_\mu A_\nu)$

Field Momentum: $\Pi^{\mu\nu} \equiv \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu} = (F^{\mu\nu} - \mathcal{U}^{\mu\nu})$

Torsion Appears: $\frac{\delta\mathcal{L}}{\delta A_\nu} = j^\nu - \frac{1}{2} \tau^\nu_{\mu\sigma} \Pi^{\mu\sigma}$

Integration by parts ($\delta A_\nu = 0$ on boundary),

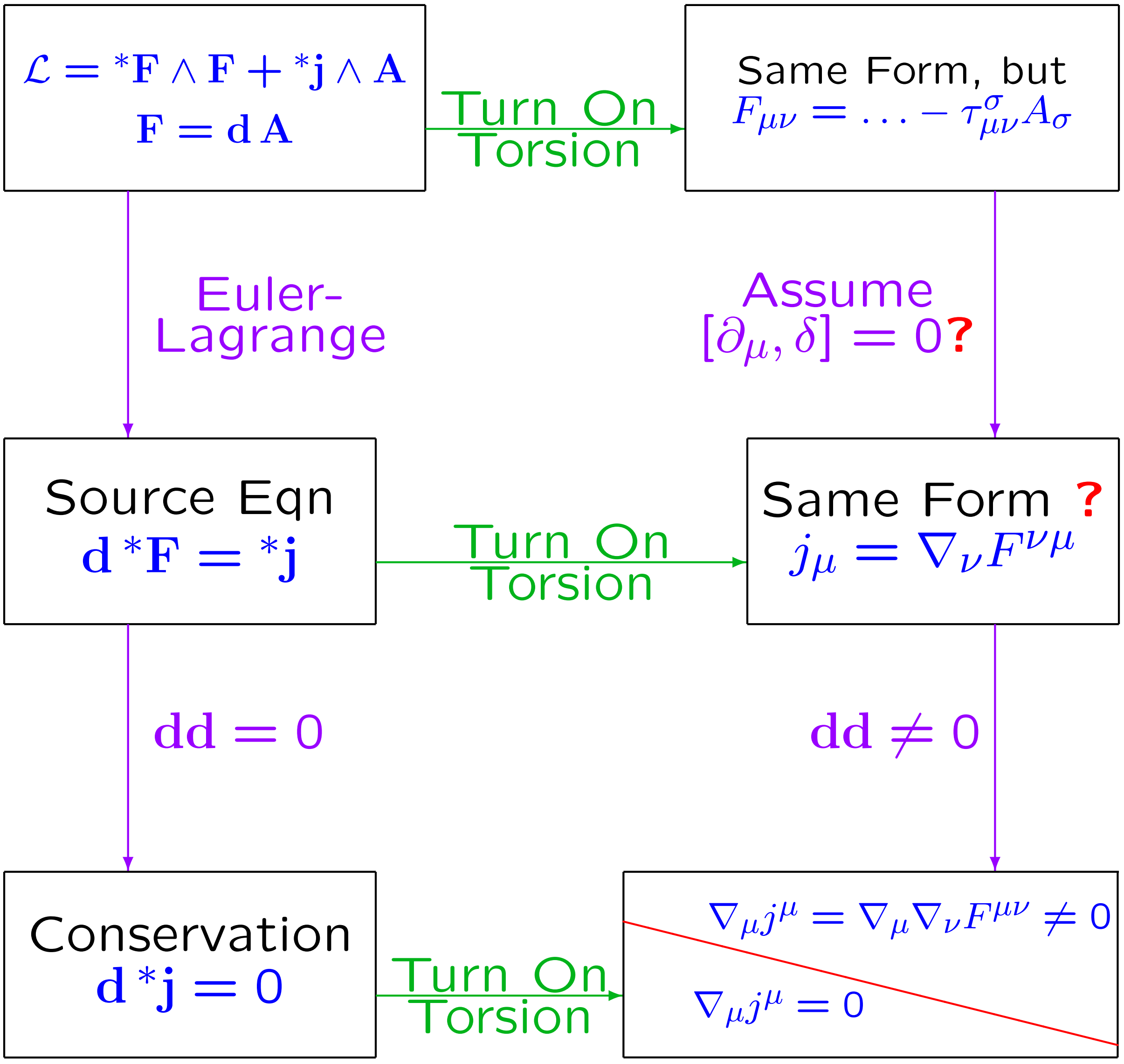
$$\Pi^{\mu\nu} \delta(\partial_\mu A_\nu) + \Pi^{\mu\nu} \left[\partial_\mu, \delta\right] A_\nu$$
$$= \left[-\nabla_\mu \Pi^{\mu\sigma} + \tfrac{1}{2} \tau^\sigma_{\mu\nu} \Pi^{\mu\nu} \right] \delta A_\sigma$$

If the commutator is zero we recover the generalized inhomogeneous equation:

$\nabla_\mu \Pi^{\mu\nu} = j^\nu$ or ${}^*\mathbf{d}\,{}^*\mathbf{F} = \mathbf{j} + {}^*\mathbf{d}\,{}^*\mathcal{U}$

But this possibly still has problems with charge conservation. Is it possible that variations do NOT commute with derivatives, such that the commutator can be used to “repair” charge conservation?

Conservation of Charge



$\mathcal{L} = {}^*\mathbf{F} \wedge \mathbf{F} + {}^*\mathbf{j} \wedge \mathbf{A}$

$\mathbf{F} = \mathbf{d}\,\mathbf{A}$

Turn On Torsion

Same Form, but

$F_{\mu\nu} = \ldots - \tau^{\sigma}_{\mu\nu} A_\sigma$

Euler-Lagrange

Assume $[\partial_\mu, \delta] = 0$**?**

Source Eqn

$\mathbf{d}\,{}^*\mathbf{F} = {}^*\mathbf{j}$

Turn On Torsion

Same Form **?**

$j_\mu = \nabla_\nu F^{\nu\mu}$

$\mathbf{dd} = 0$

$\mathbf{dd} \neq 0$

Conservation

$\mathbf{d}\,{}^*\mathbf{j} = 0$

Turn On Torsion

$\nabla_\mu j^\mu = \nabla_\mu \nabla_\nu F^{\mu\nu} \neq 0$

$\nabla_\mu j^\mu = 0$

How force "closure" of theory?

- Consider: $[\partial_\mu, \delta] A_\nu \neq 0$
- Modify Source Equation
- Modify Conservation Law?

VII. Spinning Charged Source

Particles follow paths (polygeodesics) which extremize the length swept out by the momentum ± the area swept out by the spin.[7,8]

$$L = m_o\sqrt{\overset{\circ}{x}{}^{\mu}\,\overset{\circ}{x}{}_{\mu} - \frac{1}{2\lambda^2}\,\overset{\circ}{a}{}^{\mu\nu}\,\overset{\circ}{a}{}_{\mu\nu}} \qquad \overset{\circ}{Q} \equiv \frac{dQ}{d\kappa}$$

- New Affine Parameter (based on Invariant)

$$d\kappa^2 = dx^{\mu}dx_{\mu} - \frac{1}{2\lambda^2}da^{\mu\nu}da_{\mu\nu}$$

- Fundamental length λ (radius of gyration)
- Yields Papapetrou Equations, with torsion

Interaction $$L_I = e\left(\overset{\circ}{x}{}^{\mu}A_{\mu} - \tfrac{1}{2}\,\overset{\circ}{a}{}^{\mu\nu}F_{\mu\nu}\right)$$

Current: $$j^{\sigma}(x) = \int d\kappa\, e\, \overset{\circ}{z}{}^{\sigma}\,\delta(x^{\mu} - z^{\mu}(\kappa))$$

Dipole: $$\mathcal{U}^{\mu\nu}(x) = \int d\kappa\, e\, \overset{\circ}{a}{}^{\mu\nu}\delta(x^{\mu} - z^{\mu}(\kappa))$$

Conservation of charge along worldline:[6]

$$0 = \int d\kappa\, \overset{\circ}{e}\,\delta(x^{\mu} - z^{\mu}(\kappa)) = -\int d\kappa\, e\,\frac{d}{d\kappa}\delta(x^{\mu} - z^{\mu}(\kappa))$$

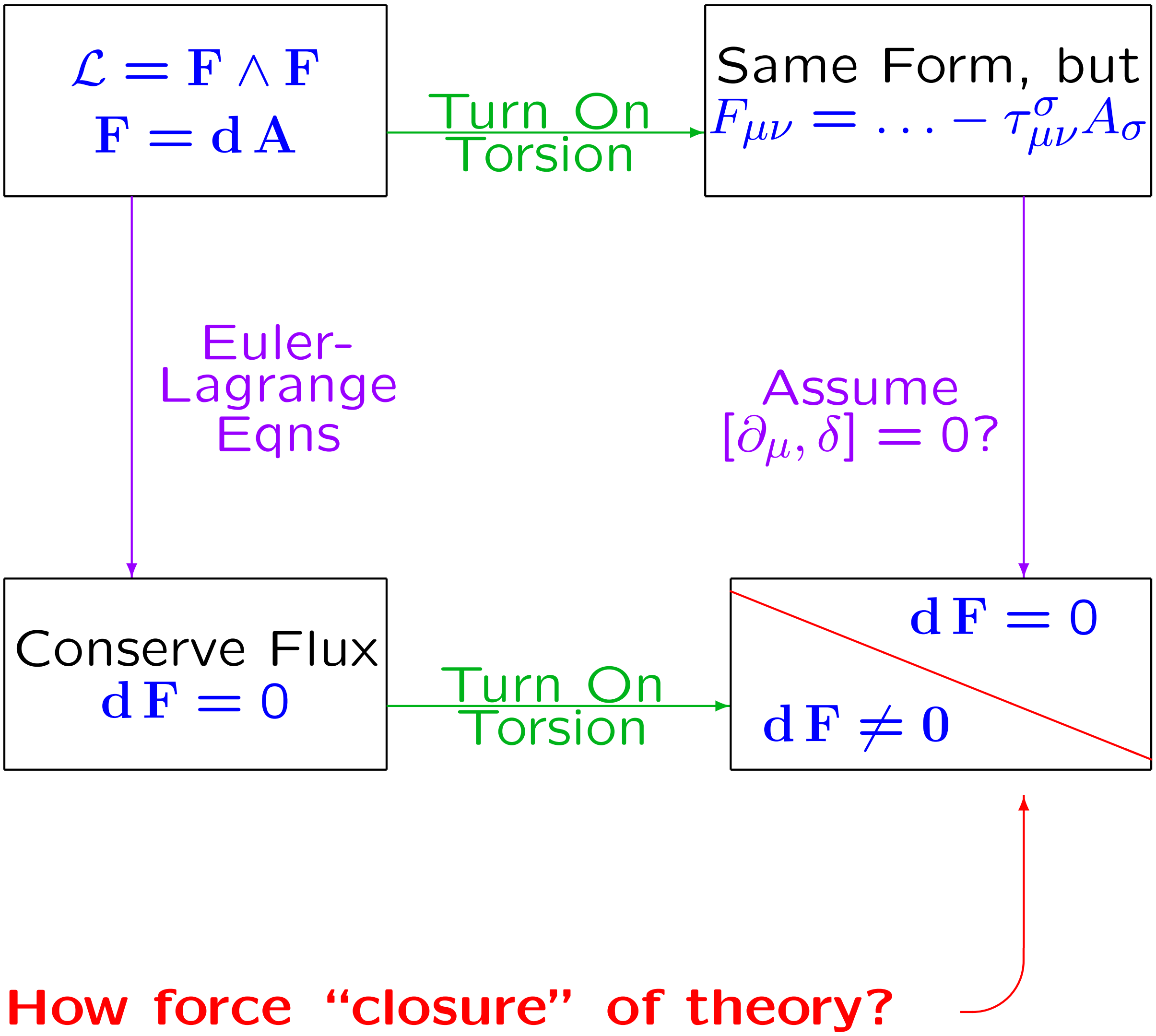
$$= \int d\kappa\, e\left[\overset{\circ}{z}{}^{\mu}\nabla_{\mu} - \tfrac{1}{2}\,\overset{\circ}{\mathcal{A}}{}^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\right]\delta(x^{\mu} - z^{\mu}(\kappa))$$

Conservation Law: $$\boxed{\nabla_{\sigma}\left(j^{\sigma} - \tfrac{1}{2}\tau^{\sigma}_{\mu\nu}\mathcal{U}^{\mu\nu}\right) = 0}$$

Can also be derived from a gauge principle $\delta A_{\mu} = \delta\nabla_{\mu}\phi$ on L_I if we argue $[\partial_{\mu}, \delta]\phi = 0$.

Dipole mimics current under torsion.

Conservation of Flux



$\mathcal{L} = \mathbf{F} \wedge \mathbf{F}$

$\mathbf{F} = \mathbf{d}\,\mathbf{A}$

Turn On Torsion

Same Form, but

$F_{\mu\nu} = \ldots - \tau^{\sigma}_{\mu\nu} A_\sigma$

Euler-Lagrange Eqns

Assume $[\partial_\mu, \delta] = 0$?

Conserve Flux

$\mathbf{d}\,\mathbf{F} = 0$

Turn On Torsion

$\mathbf{d}\,\mathbf{F} = 0$

$\mathbf{d}\,\mathbf{F} \neq \mathbf{0}$

How force “closure” of theory?

- Consider: $[\partial_\mu, \delta] A_\nu \neq 0$
- Include Magnetic Monopoles
- Include *Fluxoids*: $\mathbf{d}\,\mathbf{F} = \mathrm{d}\mathcal{U}$

VIII. Main Points

Homogeneous Lagrangian: Conserve Flux

$$\mathcal{L} = \tfrac{1}{2}\mathbf{F}\wedge\mathbf{F} + \mathbf{F}\wedge\mathcal{U}$$

Where: $\mathbf{F} = \mathbf{dA}$, $\sqcap = {}^*(\mathbf{F} - \mathcal{U})$

Without $\mathcal{U}$ inconsistent: $0 = \mathbf{dF} = \mathbf{ddA} \neq 0$

More careful analysis gives us,

$$\boxed{\delta A_\sigma \nabla_\mu \sqcap^{\mu\sigma} \;=\; \sqcap^{\mu\nu}[\partial_\mu, \delta]A_\nu \;\neq\; 0\ ?}$$

Inhomogeneous Lagrangian: Charge Consv

$$\mathcal{L} = \tfrac{1}{2}\mathbf{F}\wedge{}^*\mathbf{F} + \mathbf{F}\wedge{}^*\mathcal{U} + \mathbf{A}\wedge{}^*\mathbf{j}$$

Without $\mathcal{U}$ get source equation: $\mathbf{j} = {}^*\mathbf{d}{}^*\mathbf{F}$

Violates Charge Conservation: $\mathbf{d}{}^*\mathbf{j} = \mathbf{dd}{}^*\mathbf{F} \neq 0$

More careful analysis gives us, $\sqcap = \mathbf{F} - \mathcal{U}$

$$\boxed{\delta A_\sigma(\nabla_\mu \sqcap^{\mu\sigma} - j^\sigma) = \sqcap^{\mu\nu}[\partial_\mu, \delta]A_\nu \neq\; 0\ ?}$$

- Torsion may introduce non-commutivity of derivatives and variations
- This may help us restore conservation of flux and charge
- Torsion makes dipoles mimic charges, modifying conservation law

X. References

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