CSUEB: Matthew Spitzer Memorial Lecture Series in Physics

Friday November 8, 12:00 noon- 1:00 pm, South Science 149

What's a dark-matter pseudoscalar field anyway?

Insights using Clifford's Geometric Algebra

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Abstract

- The quantum mechanical Pauli Spin Matrices (1927) and relativistic Dirac Matrices (1928) are two common examples of Clifford algebra. Reinterpreted geometrically, these matrices represent lines, planes, volumes etc of dimensional space. For example, the "imaginary" i is concretely associated with the volume of three dimensions. A particle can be described as a multigeometric sum of its observables, e.g. energy (scalar) plus momentum (line) plus spin (plane) plus helicity (volume). Hence the work, force and torque equations can be solved in concert from a single multivector equation of motion.

- Interactions can be viewed as generalized "transdimensional" rotations that reshuffle geometry. For example, a scalar particle (e.g. Higg's boson) would have a "field" of a vector which rotates lines into planes and vice versa. Thus, one derives new forces (lines, aka "vectors") that come from the spin (plane, aka "bivector") interacting with the field, and new torques (bivectors) that come from the momentum (vector) interacting with the field.

Index

I. Introduction

1. Why we use Vectors in Physics
2. What are pseudoscalars?
3. Why Gibb's is not enough
4. Think "out of the box"

1. Why use (Gibbs') Vectors in Physics?

Example: Electromagnetism in 3D

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Coord Free

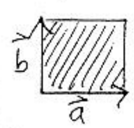
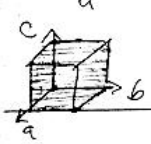
$$\begin{cases} \partial_y E_z - \partial_z E_y = -\partial_t B_x \\ \partial_z E_x - \partial_x E_z = -\partial_t B_y \\ \partial_x E_y - \partial_y E_x = -\partial_t B_z \end{cases}$$

Coordinate dependent!

- Notational Economy (3 equations in one)
- Coordinate free (physics should not depend upon coordinate system)
- Encodes isotropy of space

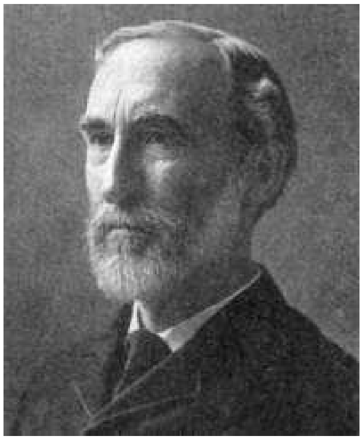
A mathematical language has utility when the metaprinciples of physics (e.g. isotropy) are built into its syntax

2. What are pseudovectors & Pseudoscalars?

Geometry	Name	Gibbs
	Point	$\lvert\vec{V}\rvert$ Scalar
	Line	$\vec{V}$ Vector
	Plane	$\vec{a} \times \vec{b}$ Pseudovector
	Volume	$\vec{a} \times \vec{b} \cdot \vec{c}$ Pseudoscalar

Gibbs' vector algebra represents a plane by the vector perpendicular to it, which means a "vector" can mean either a line or a plane (ambiguous)

3. Gibbs' Vector Inadequacies



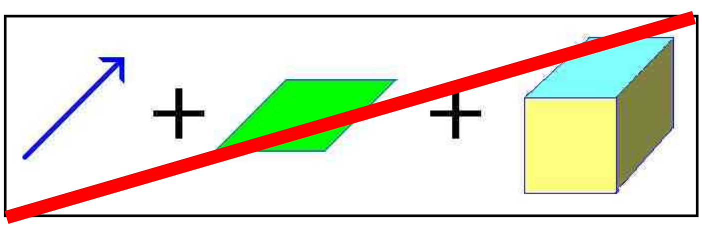
J. W. Gibbs 1839-1903
Gibbs' vectors 1881

- Can not do Higher Dimensions: Cross Product won't generalize to 4D (relativity)
- Incompleteness: there are only vectors and scalars. You can't directly represent a plane.
- Ambiguity: Given vector V=(3i+4j) does it represent a directed line (vector) or the plane perpendicular to it (pseudovector)?
- Parity Problem: the cross product is not preserved under mirror reflections (the cross product is not really a true vector, rather a "pseudovector").

"... a sort of hermaphrodite monster, compounded of the notations of Hamilton and Grassmann" -Tait

4. Thinking "out of the box"

A particular language might build in **unquestioned prejudices.**



Gibbs' algebra (and conventional tensors) have
"Dimensional Segregation",
You cannot add different ranked geometries

II. Geometric Algebra

A. Grassmann Algebra

B. Clifford's Algebra

C. Applications in Electrodynamics

D. Geometric Calculus

A **Grassmann Algebra**

Hermann Grassmann
(1809-1877)

Inventor of "Linear Algebra"

1844 publishes massive work
(which nobody understands)

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A. Grassmann Algebra

1. Extended Numbers
2. The Exterior Product
3. Hodge Dual (and other products)
4. Products of higher dimension

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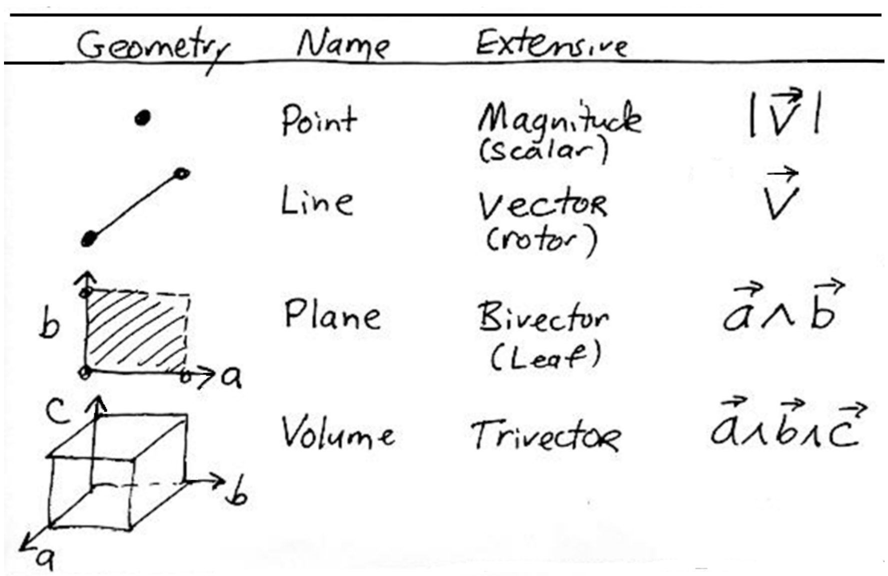
1. Extended Directed Numbers



2. The Exterior (Outer) Product

Antisymmetric: $\vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a}$

Neutral Algebra: $\vec{a} \wedge \vec{a} = 0$

Associative: $\left(\vec{a} \wedge \vec{b}\right) \wedge \vec{c} = \vec{a} \wedge \left(\vec{b} \wedge \vec{c}\right)$

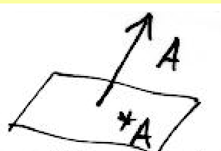
Closed (e.g. in 3D): $\vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} = 0$

3. The Dual and Inner (dot) Product

Hodge geometric dual

$^*A$ = dual of A

$^{**}A = -A$



in 3D dual of vector is plane

Dot Product: $\vec{a} \bullet \vec{b} \equiv -{}^*\left(\vec{a} \wedge {}^*\vec{b}\right)$

Cross Product: $\vec{a} \times \vec{b} \equiv -{}^*\left(\vec{a} \wedge \vec{b}\right)$

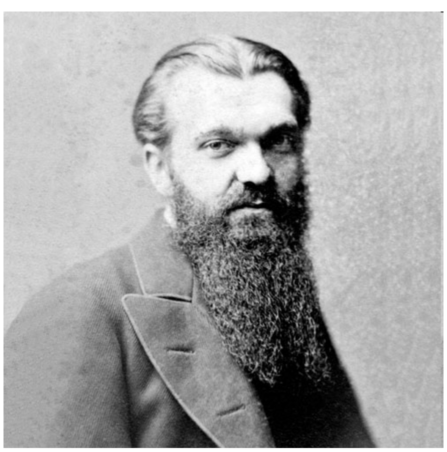
4. Products of Planes and Lines

$$\vec{A}\bullet\left(\vec{B}\wedge\vec{C}\right)=+\left(\vec{A}\bullet\vec{B}\right)\vec{C}-\left(\vec{A}\bullet\vec{C}\right)\vec{B}$$

$$\vec{A}\bullet\left(\vec{B}\wedge\vec{C}\wedge\vec{D}\right)=+\left(\vec{A}\bullet\vec{B}\right)\vec{C}\wedge\vec{D}-\left(\vec{A}\bullet\vec{C}\right)\vec{B}\wedge\vec{D}+\left(\vec{A}\bullet\vec{D}\right)\vec{B}\wedge\vec{C}$$

No 4D Object

B. Clifford Algebra



William Kingdon Clifford (1845-1879)

- Translated Riemann's work
- Anticipated general relativity
- Died shortly after inventing algebra which combined Hamilton's and Grassmann's ideas into one form
- Forgotten until recently.

B. Clifford Algebra

1. Definition of Clifford Algebra
2. The 3D Pauli Algebra
3. Geometric “direct” product of vectors

1. Defining a Clifford Algebra

For “N” dimensions
have “N” basis vectors $\{\sigma_j\} \quad j = 1, \cdots, N$

•They anticommute $\sigma_1\sigma_2 = -\sigma_1\sigma_2$

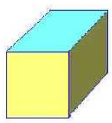
•Square to +1 $\sigma_1\sigma_1 = \sigma_2\sigma_2 = +1$

“Anticommutivity is perpendicularity”

Or: single rule

$$2\delta_{ij} = \{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_i\sigma_j$$

2a. 3D Clifford Algebra is Pauli Algebra

Geometry	Name	Pauli Element	Spin^{Parity}
•	Scalar	1	0^+
	Vector	σ_1 σ_2 σ_3	1^-
	Bivector	$\sigma_2\sigma_3 = i\sigma_1$ $\sigma_3\sigma_1 = i\sigma_2$ $\sigma_1\sigma_2 = i\sigma_3$	1^+
	Trivector	$\sigma_1\sigma_2\sigma_3 = i$	0^-

*Geometric interpretation of "**i**" is volume*
*Multiplying by "**i**" gives the dual*

2b Matrix Representation

Pauli algebra can be **represented** by 2x2 complex matrices: C(2)

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2c Hermetian Conjugate

- The conjugate is an "anti-semilinear involution" (reverses order)

$$(ab)^\dagger = b^\dagger a^\dagger$$

- In matrix form it is the complex conjugate of the transpose.

$$i^\dagger = -i$$

$$\sigma_1^\dagger = +\sigma_1$$

$$(i\sigma_3)^\dagger = (\sigma_1\sigma_2)^\dagger = (\sigma_2\sigma_1) = -i\sigma_3$$

3. Geometric (direct) product of vectors is a multivector

- Consider two vectors

$\vec{A} = a_1\sigma_1 + a_2\sigma_2$ $\quad$ $\vec{B} = b_1\sigma_1 + b_2\sigma_2$

- Multiply them

$$\vec{A}\vec{B} = (a_1\sigma_1 + a_2\sigma_2)(b_1\sigma_1 + b_2\sigma_2)$$
$$= (a_1b_1 + a_2b_2) + (a_1b_2 - a_2b_1)\sigma_1\sigma_2$$

- The product is a SCALAR + BIVECTOR:

$$\vec{A}\vec{B} = +\vec{A} \bullet \vec{B} + \vec{A} \wedge \vec{B}$$

3b Definition of Products

- Product
- Reverse

$$\vec{A}\vec{B} = +\vec{A} \bullet \vec{B} + \vec{A} \wedge \vec{B}$$

$$\vec{B}\vec{A} = +\vec{A} \bullet \vec{B} + \vec{B} \wedge \vec{A}$$

- Dot product is symmetric interchange

$$\vec{A} \bullet \vec{B} = +\vec{B} \bullet \vec{A} = \tfrac{1}{2}\left(AB + BA\right) = \tfrac{1}{2}\{A, B\}$$

- Wedge product is antisymmetric interchange

$$\vec{A} \wedge \vec{B} = -\vec{B} \wedge \vec{A} = \tfrac{1}{2}\left(AB - BA\right) = \tfrac{1}{2}[A, B]$$

C. Multivector Electrodynamics

1. Define the **Electromagnetic Field Multivector** (sum of vector electric field and bivector magnetic field)

$$\begin{aligned} F &= \vec{E} + i\vec{B} \\ &= E_x\sigma_1 + E_y\sigma_2 + E_z\sigma_3 \\ &\quad + B_x\sigma_2\sigma_3 + B_y\sigma_3\sigma_1 + B_z\sigma_1\sigma_2 \end{aligned}$$

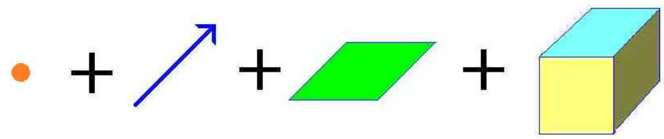
The conjugate: $F^\dagger = \vec{E} - i\vec{B}$

2. Dimensional Democracy

Define the "Polymomena" as the multivector of the energy, momentum, spin and helicity

$$M = \tfrac{1}{c}\xi + \vec{P} + i\vec{S} + \tfrac{1}{c}iH$$

Scalar + Vector + Bivector + Trivector



3 Multivector Equation of Motion

$$\dot{M} = \tfrac{e}{2mc}\left(MF^{\dagger} + FM\right)$$

Four equations of motion in one!

$$\begin{cases} Scalar: & \dot{\xi} = \tfrac{e}{m}\vec{P}\bullet\vec{E} \\ Vector: & \dot{\vec{P}} = e\left(\vec{E} + \vec{v}\times\vec{B}\right) \\ Bivector: & i\dot{\vec{S}} = i\tfrac{e}{m}\left(H\vec{E} + \vec{S}\times\vec{B}\right) \\ Trivector: & i\dot{H} = i\tfrac{e}{m}\vec{S}\bullet\vec{E} \end{cases}$$

4 Solution

Can solve all four equations simultaneously in this multivector form!

$$\dot{M} = \tfrac{e}{2mc}\left(MF^{\dagger} + FM\right)$$

$$M(t) = R\,M_0\,R^{\dagger}$$

$$R(t) = \exp\left(\frac{e}{2mc}\int_0^t F\,dt\right)$$

$$M_0 = M(0)$$

http://www.youtube.com/watch?feature=player_detailpage&v=1eOn5K9qZu4

D. Geometric Calculus

1. Derivative of Vector is a multivector sum of scalar and bivector:

$$\nabla\vec{E} = \nabla \bullet \vec{E} + \nabla \wedge \vec{E}$$

Derivative multivector field:

$$\nabla F = \nabla\left(\vec{E} + i\vec{B}\right)$$

$$= \nabla \bullet \vec{E} + \nabla \bullet i\vec{B} + \nabla \wedge \vec{E} + \nabla \wedge i\vec{B}$$

$$= \nabla \bullet \vec{E} - \nabla \times \vec{B} + i\nabla \times \vec{E} + i\nabla \bullet \vec{B}$$

scalar *vector* *bivector* *trivector*

2. Multivector Field Equation

You can get 4 Maxwell's equations in ONE!

$$\left(\partial_t + \vec{\nabla}\right)F = \left(\rho - \vec{J}\right) \qquad F = \vec{E} + i\vec{B}$$

$$= \begin{Bmatrix} \nabla \bullet \vec{E} = \rho & scalar \\ -\nabla \times \vec{B} + \frac{1}{c}\dot{E} = - + \frac{1}{c}\vec{J} & vector \\ i\left(\nabla \times \vec{E} + \frac{1}{c}\dot{B}\right) = 0 & bivector \\ i\nabla \bullet \vec{B} = 0 & trivector \end{Bmatrix}$$

Jancewicz, **Multivectors and Clifford Algebras in Electrodynamics**, (1988 World Scientific) pg. 78

3. Example of Utility of Clifford Algebra

(4.18) $\hat{E}(x^1, x^2, x^3) = E(h, x^1, x^2, x^3)$

(4.19) $\hat{H}(x^1, x^2, x^3) = H(h, x^1, x^2, x^3)$

where h is the value of x^0 given by (4.17). The vector functions $\hat{E}$ and $\hat{H}$ are assumed to have continuous first derivatives. Using the above definitions and a bit of vector algebra, we shall be able to obtain a pair of very useful relations (V. Fock, 1959) which the functions E, H, $\hat{E}$, $\hat{H}$, and h must obey on S. These relations will be the key to obtaining the characteristic surfaces of Maxwell's equations.

From (4.18), (4.19), and (4.17) we have

(4.20) $\frac{\partial \hat{E}_k}{\partial x^i} = \frac{\partial E_k}{\partial x^i} + \frac{\partial E_k}{\partial x^0}\frac{\partial h}{\partial x^i}$

Setting $k = i$ and summing from 1 to 3, we obtain

(4.21) $\nabla \cdot \hat{E} = \nabla \cdot E + \frac{1}{c}\dot{E} \cdot \nabla h$

$\nabla \cdot \hat{H} = \nabla \cdot H + \frac{1}{c}\dot{H} \cdot \nabla h$

From (4.20) we also obtain

$\frac{\partial E_k}{\partial x^i} - \frac{\partial E_i}{\partial x^k} + \frac{\partial E_k}{\partial x^0}\frac{\partial h}{\partial x^i} - \frac{\partial E_i}{\partial x^0}\frac{\partial h}{\partial x^k} = \frac{\partial \hat{E}_k}{\partial x^i} - \frac{\partial \hat{E}_i}{\partial x^k}$

that is, in vector notation,

(4.24) $\nabla \times E + \frac{1}{c}\nabla h \times \dot{E} = \nabla \times \hat{E}$

and as the analogous result for the magnetic field, we have

(4.25) $\nabla \times H + \frac{1}{h}\nabla h \times \dot{H} = \nabla \times \hat{H}$

$F = E + iH$

$\hat{F} = F(h(r), \vec{r})$

by chain rule

$\nabla \hat{F} = \nabla F + \frac{1}{c}\nabla h\left(\partial_t F\right)$

substitute Maxwell's Eqns in vac

(4.12) $\nabla \cdot E = 0$

(4.11) $\nabla \times H = \frac{1}{c}\dot{E}$

(4.13) $\nabla \times E = -\frac{1}{c}\dot{H}$

(4.14) $\nabla \cdot H = 0$

$\nabla F = -\frac{1}{c}\partial_t F$

Left side is using Gibbs vectors, right side using Clifford Algebra

Continued

From (4.12) this gives

$$\nabla \cdot \hat{E} = \frac{1}{c}\dot{E} \cdot \nabla h \qquad (4.22)$$

For the magnetic field we may obtain the analogous result

$$\nabla \cdot \hat{H} = \frac{1}{c}\dot{H} \cdot \nabla h \qquad (4.23)$$

Substituting into these last two equations from Maxwell's equations in vacuum, (4.11) and (4.13), we obtain

$$-\frac{1}{c}\dot{H} + \frac{1}{c}\nabla h \times \dot{E} = \nabla \times \hat{E} \qquad (4.26)$$

$$\frac{1}{c}\dot{E} + \frac{1}{c}\nabla h \times \dot{H} = \nabla \times \hat{H} \qquad (4.27)$$

$$\nabla \hat{F} = (\nabla h - 1)\dot{F}$$

The scalar product of (4.26) and (4.27) with ∇h gives

$$-\frac{1}{c}\nabla h \cdot \dot{H} + \frac{1}{c}\nabla h \cdot \nabla h \times \dot{E} = -\frac{1}{c}\nabla h \cdot \dot{H} = \nabla h \cdot \nabla \times \hat{E} \qquad (4.28)$$

$$\frac{1}{c}\nabla h \cdot \dot{E} + \frac{1}{c}\nabla h \cdot \nabla h \times \dot{H} = \frac{1}{c}\nabla h \cdot \dot{E} = \nabla h \cdot \nabla \times \hat{H} \qquad (4.29)$$

To scalarize the right side, multiply on left by $(\nabla h + 1)$

Also the vector product of (4.26) and (4.27) with ∇h gives

$$-\frac{1}{c}(\nabla h \times \dot{H}) + \frac{1}{c}\nabla h \times (\nabla h \times \dot{E}) = \nabla h \times (\nabla \times \hat{E}) \qquad (4.30)$$

$$\frac{1}{c}(\nabla h \times \dot{E}) + \frac{1}{c}\nabla h \times (\nabla h \times \dot{H}) = \nabla h \times (\nabla \times \hat{H}) \qquad (4.31)$$

Expanding the double cross product and substituting from (4.26) and (4.27), we have

$$\frac{1}{c}\dot{E} - \nabla \times \hat{H} + \frac{1}{c}\nabla h(\nabla h \cdot \dot{E}) - \frac{1}{c}\dot{E}(\nabla h)^2 = \nabla h \times (\nabla \times \hat{E}) \qquad (4.32)$$

$$\frac{1}{c}\dot{H} + \nabla \times \hat{E} + \frac{1}{c}\nabla h(\nabla h \cdot \dot{H}) - \frac{1}{c}\dot{H}(\nabla h)^2 = \nabla h \times (\nabla \times \hat{H}) \qquad (4.33)$$

Finally, substituting from (4.28) and (4.29), we get

$$\frac{1}{c}\dot{E} - \nabla \times \hat{H} + \nabla h(\nabla h \cdot \nabla \times \hat{H}) - \frac{1}{c}\dot{E}(\nabla h)^2 = \nabla h \times (\nabla \times \hat{E}) \qquad (4.34)$$

$$\frac{1}{c}\dot{H} + \nabla \times \hat{E} - \nabla h(\nabla h \cdot \nabla \times \hat{E}) - \frac{1}{c}\dot{H}(\nabla h)^2 = \nabla h \times (\nabla \times \hat{H}) \qquad (4.35)$$

Rearrangement now gives the two key relations that we have been working toward:

$$\frac{1}{c}(1 - |\nabla h|^2)\dot{E} = \nabla \times \hat{H} - \nabla h(\nabla h \cdot \nabla \times \hat{H}) + \nabla h \times (\nabla \times \hat{E}) \qquad (4.36)$$

$$\frac{1}{c}(1 - |\nabla h|^2)\dot{H} = -\nabla \times \hat{E} + \nabla h(\nabla h \cdot \nabla \times \hat{E}) + \nabla h \times (\nabla \times \hat{H}) \qquad (4.37)$$

$$0 = \nabla \cdot \hat{E} - \nabla h \cdot (\nabla \times \hat{H})$$

$$0 = \nabla \cdot \hat{H} + \nabla h \cdot (\nabla \times \hat{E})$$

$$(\nabla h + 1)\nabla \hat{F} = ((\nabla h)^2 - 1)\dot{F}$$

Solving 4 equations in one can save many steps!

III. Spacetime Geometry

A. Algebra of 4D

B. Electrodynamics in 4D

C. Rotations and Lorentz Transformations

A. Four Dimensional Spacetime

1. Time is the 4th dimension
2. Clifford algebra of 4D
3. Matrix Representation

1. Metric of Time

- Euclidean space obeys the Pythagorean theorem. Measurement of length:

$$dr = \sqrt{dx^2 + dy^2 + dz^2}$$

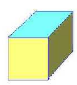
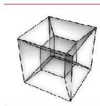
- A spacetime "length" has the metric of time being negative relative to the space portion

$$ds = \sqrt{dr^2 - c^2 dt^2}$$

- Definition of "proper time" $d\tau$

$$d\tau = \sqrt{dt^2 - \tfrac{1}{c^2} dr^2}$$

2. 4D Clifford algebra is Majorana Algebra

Geometry	Name	Element	Metric
•	Scalar	1	$+1$
	Vector	e_1	$+1$
		e_2	$+1$
		e_3	$+1$
		e_4	-1
	Bivector	e_4e_1	$+1$
		e_4e_2	$+1$
		e_4e_3	$+1$
		e_1e_2	-1
		e_2e_3	-1
		e_3e_1	-1
	Trivector (Pseudovector)	$e_2e_3e_4 = -\varepsilon e_1$	$+1$
		$e_1e_3e_4 = -\varepsilon e_2$	$+1$
		$e_1e_2e_4 = -\varepsilon e_3$	$+1$
		$e_1e_2e_3 = -\varepsilon e_4$	-1
	Quadvector (Pseudoscalar)	$e_1e_2e_3e_4 = \varepsilon$	-1

There are two different "metric signatures" that work for special relativity: (- - - +) or (+ + + -)

Note pseudoscalar does NOT commute with vectors in even dimensions

3 Matrix Representation

Majorana algebra can be **represented** by 4x4 real matrices: R(4)

$$e_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad e_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

B. Example: Electrodynamics in 4D

1. Unify Phenomena with 4th Dimension

Combine scalar law with vector law using 4D vectors

Scalar $\dot{\xi} = e\vec{E} \bullet \vec{v}$

Vector $\dot{\vec{P}} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$

$$\dot{p} = \tfrac{e}{m}\, p \bullet F$$

4 vector: $\mathbf{p} = p^1\hat{e}_1 + p^2\hat{e}_2 + p^2\hat{e}_3 + (E/c)\hat{e}_4$

Field Bivector $F = E^1\hat{e}_4\wedge\hat{e}_1 + E^2\hat{e}_4\wedge\hat{e}_2 + E^3\hat{e}_4\wedge\hat{e}_3 + B^1\hat{e}_2\wedge\hat{e}_3 + B^2\hat{e}_3\wedge\hat{e}_1 + B^3\hat{e}_1\wedge\hat{e}_2$

2. Unify Phenomena Dimensionally

Using Clifford Algebra, get 2 equations in 1

Line: $\dot{p}^\mu = \frac{e}{m} p_\nu F^\mu{}_\nu$

Plane: $\dot{S}^{\mu\nu} = \frac{e}{m}\left(F^\mu{}_\sigma S^{\sigma\nu} - F^\nu{}_\sigma S^{\sigma\mu}\right)$

$$\dot{\mathcal{M}} = \frac{e}{2m}[\mathcal{M}, \mathbf{F}]$$

Polymomenta: $\mathcal{M} \equiv \underbrace{p^\mu\, \hat{\mathbf{e}}_\mu}_{Line} + \underbrace{\tfrac{1}{2} S^{\mu\nu}\, \hat{\mathbf{e}}_\mu \wedge \hat{\mathbf{e}}_\nu}_{Plane}$

P is the momentum, **S** is the spin, **F** is the electromagnetic field

3 Solution

Again, solve for momentum and spin in one step.

$$\dot{M} = \tfrac{e}{2mc}\left[M, F\right]$$

$$M(\tau) = R\, M_0\, R^{-1}$$

$$R(\tau) = \exp\left(\tfrac{e}{4mc}\, \hat{e}_\mu \wedge \hat{e}_\nu \int_0^\tau F^{\mu\nu}\, d\tau \right)$$

The nature of "R" is that it **rotates** the "poly-momentum" in the **plane** of the **field bivector**

C. Rotations

1. Rotation generated by a bivector
2. Lorentz Transformation
3. Properties of Rotations

1. Clifford Algebra Form of Rotation

- Rotation of any quantity "Q" in "xy" plane by angle ϕ can be expressed in exponential "half angle" form:

$$Q' = R\,Q\,R^{-1}$$

$$R = e^{\hat{e}_1\hat{e}_2\phi/2} = \cos\tfrac{\phi}{2} + \hat{e}_1\hat{e}_2 \sin\tfrac{\phi}{2}$$

Note "spacelike" bivectors square negative: $(\hat{e}_1\hat{e}_2)^2 = -1$

The Magnetic field "rotates" the direction of motion of particle

2. Lorentz Transformation

- Hyperbolic Rotation in "TX" plane by rapidity angle β has same exponential "half angle" form:

$$Q' = L\,Q\,L^{-1}$$

$$L = e^{\hat{e}_4\hat{e}_1\beta/2} = \cosh\tfrac{\beta}{2} + \hat{e}_4\hat{e}_1 \sinh\tfrac{\beta}{2}$$

Note "Timelike" bivectors square positive: $(\hat{e}_4\hat{e}_1)^2 = +1$
because

$$(\hat{e}_4)^2 = -1$$

$$(\hat{e}_1)^2 = +1$$

The Electric field "rotates" the particle in spacetime, i.e. acceleration

3. Properties of Rotations

- **<u>Isometry</u>: preserves modulus specifically length of 4-vector:** $p^\mu p_\mu = -m^2$
- **Preserves <u>rank</u> of geometry (vectors rotated into vectors)**
- **Preserves Clifford product (algebra automorphism)**

$$(AB)' = A'B'$$

$$R(AB)R^{-1} = \left(RAR^{-1}\right)\left(RBR^{-1}\right)$$

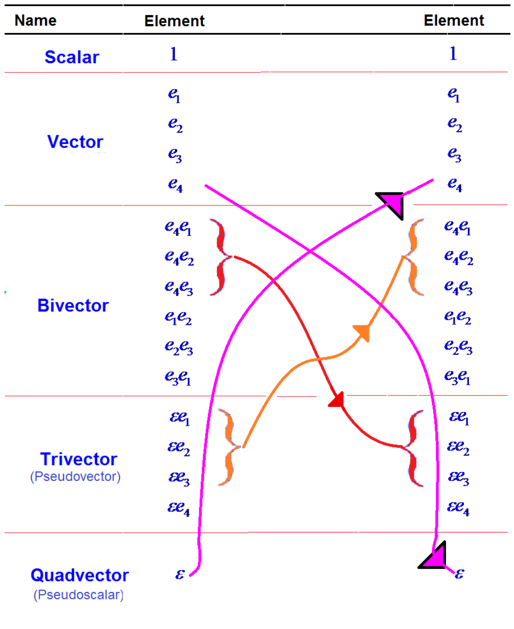
IV. Transdimensional Physics

A. Automorphism Invariance

B. Pseudoscalar has trivector field

C. New macroscopic forces and torques?

A. Automorphism Invariance



Automorphism is an invertible transformation that preserves the Clifford algebra structure

$$2g_{\mu\nu} = \{e_\mu, e_\nu\}$$

Example: “rotation” by trivector $\boldsymbol{\varepsilon e_4}$

$$R = e^{\hat{\varepsilon}\hat{e}_4\theta/2}$$

Suggests dimension is relative to observer!

B. Other Fields

POTENTIAL		FIELD	
Scalar	ψ	$\Box\psi = e^\mu \partial_\mu \psi$	*Vector*
Vector	$A^\mu e_\mu$	$F = \Box \wedge A$	*Bivector*
Pseudoscalar	$\hat{\varepsilon}\phi$	$\hat{\varepsilon}\Box\psi = \hat{\varepsilon}e^\mu \partial_\mu \psi$	*Trivector*

Can justify this from Gauge Theory

Gauge Covariant Derivative $\nabla_\mu = \partial_\mu + \frac{1}{4}\hat{\varepsilon}\hat{e}_\mu\phi$

Field $\hat{e}^\mu \wedge \hat{e}^\nu [\nabla_\mu, \nabla_\nu] = \hat{e}^\mu \wedge \hat{e}^\nu [\partial_\mu, \frac{1}{4}\hat{\varepsilon}\hat{e}_\nu\phi]$

B.2 Equation of Motion with Trivector Field

Propose "Trivector" field *"W"* rotates polymomenta *"M"* by the same equation*

$$\dot{M} = \tfrac{g}{2mc}\left[\hat{\varepsilon}\vec{W}, M\right] \quad , \qquad W_\mu = \partial_\mu \phi$$

$$M(\tau) = R M_0 R^{-1}$$

$$R(\tau) = \exp\left(\tfrac{g}{2mc}\, \hat{\varepsilon}\hat{e}_\mu \int_0^\tau W^\mu \, d\tau \right)$$

**I can justify this from Dirac Equation.*
"g" is coupling constant

B.3 The PolyMomenta in 4D

Polymomenta must contain more pieces since field will "rotate" between geometries

$$M = p^\mu \hat{e}_\mu + \tfrac{1}{2} S^{\mu\nu}\, \hat{e}_\mu \wedge \hat{e}_\nu + k^\mu \hat{\varepsilon}\hat{e}_\mu + \pi \hat{\varepsilon}$$

Note I can justify this from Dirac theory. Given Dirac Bispinor ψ *:* $\psi\overline{\psi} = M$

$$p_\mu \approx i\overline{\psi}\gamma_\mu\psi$$

$$S_{\mu\nu} \approx i\overline{\psi}\gamma_\mu\gamma_\nu\psi$$

$$k_\mu \approx i\overline{\psi}\gamma_5\gamma_\mu\psi$$

$$\pi \approx i\overline{\psi}\gamma_5\psi$$

C. Testable Physics?

Four equations in one: $\dot{M} = \frac{g}{2mc}\left[\hat{\varepsilon}\vec{W}, M\right]$

$$\begin{cases} \textit{Vector}: & \dot{p}_\mu = \frac{g}{2mc}\pi W_\mu \\ \textit{Bivector}: & \dot{S}_{\mu\nu} = \frac{g}{2mc}\left(\vec{W}\wedge\vec{k}\right)_{\mu\nu} \\ \textit{Trivector}: & \dot{k}_\nu = \frac{g}{2mc}W^\mu S_{\mu\nu} \\ \textit{Pseudoscalar}: & \dot{\pi} = p^\mu W_\mu \end{cases}$$

C.2. New Invariants

Mass is no longer invariant. Have:

$$p^\mu \dot{p}_\mu = \pi\, \vec{p} \bullet \vec{W} = \pi\dot{\pi}$$

$$m^2 + \pi^2 = \text{constant}$$

Spin is no longer invariant. Have:

$$S^{\mu\nu}\dot{S}_{\mu\nu} = S^{\mu\nu}W_\mu k_\nu = k_\nu \dot{k}^\nu$$

$$\tfrac{1}{2}S^{\mu\nu}S_{\mu\nu} - k_\nu k^\nu = \text{constant}$$

These may be further constrained by the Fierz identities. See Crawford, J. Math. Phys. 26 1439-1441 (1985)

C.3. Dirac Equation Details

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Can derive more correct set of equations from a Dirac Equation (here e_v are equivalent to the Dirac Gamma Matrices):

$$\hat{e}^{\mu}\nabla_{\mu}\Psi = m\Psi$$

$$\nabla_{\mu} = \partial_{\mu} + \tfrac{g}{4\hbar mc}\hat{\varepsilon}\hat{e}_{\mu}\phi$$

$$\partial_{\sigma}\left[\left(\nabla^{\sigma}\Psi\right)\overline{\Psi} - \Psi\left(\overline{\Psi}\overline{\nabla}^{\sigma}\right)\right] = \tfrac{g}{2\hbar mc}\left[\hat{\varepsilon}\hat{e}_{\mu}\partial^{\mu}\phi, \Psi\overline{\Psi}\right]$$

V. Summary:

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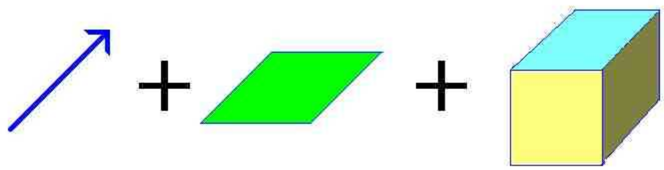
The Pezzaglia Polydimensional Principles:

Relative Dimensionalism: The geometric rank an observer assigns to an object is a function of the observer's frame, there is no "absolute" dimension.	**Polydimensional Isotropy:** There is no absolute or preferred "direction" in the universe to which one can assign the geometry of a vector.
Dimensional Democracy: Every geometric has an associated coordinate. Laws should be multivectorial (e.g. vector +bivector) with each geometry physically realized.	**Metamorphic Covariance:** The laws of physics should be form invariant under local automorphism transformations which reshuffle the physical geometry (general curvature).

V. William Kingdom Clifford (1876)

1. *That small portions of space are in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them.*
2. *That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave*
3. *That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter, whether ponderable or ethereal*
4. *That in the physical world, nothing else takes place but this variation, subject (possibly) to the law of continuity.*

Clifford Algebra has "**Dimensional Democracy**", allowing you to add lines to planes etc



References

- EM in one equation, see Bernard Jancewicz, Multivectors and Clifford Algebra in Electrodynamics (World Scientific1988) p. 78
- William Baylis, Electrodynamics, a Modern Geometric Approach (Birkhauser1999)
- The best quick introduction to Clifford Algebra is David Hestenes:
  - Space-time Algebra (Gordon & Breach 1966)
  - New Foundations for Classical Mechanics (Kluwer 1986)
  - Or, my Ph.D. thesis (I'll post this on web site)
- The best explanations of Grassmann algebra are usually in the books on Clifford Algebras. However, a standard is Flanders, Differential Forms, Academic Press (1963).
- Another summary of Grassmann algebra would be:
  D. Fearnley-Sander, American Mathematical Monthly, **86**, 809 (1979)
- W. Pezzaglia, July 29, 1996 (unpublished) wrote out the "multivector electrodynamic" equation. Other authors solve for the energy and momentum part, but do not include the spin.